On long time existence for small solutions of semi-linear Klein-Gordon equations on the torus

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Abstract

We prove that small smooth solutions of weakly semi-linear Klein-Gordon equations on the torus $\mathbb{T}^d$ ($d \geq 2$) exist over a larger time interval than the one given by local existence theory, for almost every value of the mass. We use a normal form method for the Sobolev energy of the solution. The difficulty, in comparison with previous results obtained on the sphere, comes from the fact that the set of differences of eigenvalues of $\sqrt{-\Delta}$ on $\mathbb{T}^d$ ($d \geq 2$) is dense in $\mathbb{R}$.

0 Introduction

The aim of this paper is to study long-time existence problems for semi-linear Klein-Gordon equations of type

$$
\begin{aligned}
&(\partial_t^2 - \Delta + m^2)v = F(x, v) \\
&v|_{t=0} = \epsilon v_0 \\
&\partial_t v|_{t=0} = \epsilon v_1
\end{aligned}
\tag{0.0.1}
$$

on the torus $\mathbb{T}^d$ ($d \geq 1$). If v_0, v_1 are smooth on $\mathbb{T}^d$ and if F is a smooth function vanishing at some order $\kappa + 1$ at $v = 0$, local existence theory implies that (0.0.1) admits, for small $\epsilon > 0$, a unique smooth solution defined on intervals of length $c\epsilon^{-\kappa}$. Our goal is to show that for m outside an exceptional subset of zero measure, the solution actually extends at least over an interval of length $c\epsilon^{-\kappa\left(1+\frac{2}{d}\right)}|\log \epsilon|^{-A}$, where $A > 1$ is a constant. In other words, we want to go beyond the existence time given by local existence theory, in spite of the fact that on the compact manifold $\mathbb{T}^d$ we cannot use any dispersive property of the equation.

This problem has been studied in dimension 1 by Bourgain [5], Bambusi [1], Bambusi-Grébert [3]. They showed that one has then almost global existence: for any N, if the data are in $H^{s+1} \times H^s$

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for some s depending on N, if m stays outside an exceptional subset of zero measure, the solution exists at least on an interval of length $c_N\epsilon^{-N}$. These results have been extended to equation (0.0.1) on the sphere $\mathbb{S}^d$ by Bambusi, Delort, Grébert and Szeftel [2]. The method used in that paper was combining the fact that (0.0.1) may be written as a Hamiltonian equation, together with methods developed in [8, 9] to study (0.0.1) on the sphere, for nonlinearities depending not only on v, but also on $\partial_t v, \partial_x v$.

On the other hand, the only result which was known up to now on tori of dimension at least 2 was limited to nonlinearities vanishing at the origin at order $\kappa+1=2$, and was obtained in [8]. This result was giving a solution defined on an interval of length $c\epsilon^{-2}$. The proof was relying in an essential way on the fact that the lower order term in the nonlinearity is quadratic, as we shall recall below. Let us remind also that a lot of work has been devoted to the quite different problem of construction of periodic or quasi-periodic solutions for equation (0.0.1) on $\mathbb{T}^d$. We refer to the books of Craig [7] and Bourgain [6] for results of that kind, and for a complete bibliography on such a question.

Let us explain our method, assuming for a while that we study (0.0.1) not just on the torus, but on some compact manifold M. We want to control the Sobolev energy of the solutions computing

$$\frac{d}{dt}[\|v(t,\cdot)\|^2_{H^{s+1}} + \|\partial_t v(t,\cdot)\|^2_{H^s}]. \tag{0.0.2}$$

This quantity may be written, using the equation, as a sum of multilinear expressions in $v, \partial_t v$, homogeneous of degree at least $\kappa+2$. One then tries to perturb the Sobolev energy by expressions homogeneous of degree at least $\kappa+2$ such that their times derivatives cancel out the main contribution in (0.0.2), up to remainders of higher order. The difficulty is to construct these perturbations in such a way that they will be bounded by powers of $\|v\|_{H^{s+1}} + \|\partial_t v\|_{H^s}$, with the same s as in (0.0.2). Using expansion of elements of H^s on a basis of L^2 made of eigenfunctions of $\sqrt{-\Delta}$, one is reduced to the study of expressions of type

$$\sum_{n_0,\dots,n_{p+1}} F_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})^{-1}\int_M (\Pi_{\lambda_{n_0}}u_0)\cdots(\Pi_{\lambda_{n_{p+1}}}u_{p+1})\,dx(\lambda_{n_0}+\cdots+\lambda_{n_{p+1}})^{2s} \tag{0.0.3}$$

where λ_{n_j} are eigenvalues of $\sqrt{-\Delta}$ on the compact manifold M, Π_λ is the spectral projector associated to the eigenvalue λ, and F_m is given by

$$F_m(\xi_0,\dots,\xi_{p+1}) = \sum_{j=0}^{\ell}\sqrt{m^2+\xi_j^2} - \sum_{j=\ell+1}^{p+1}\sqrt{m^2+\xi_j^2} \tag{0.0.4}$$

for some ℓ between 0 and $p+1$. The problem is to bound $|F_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})|$ from below, for those λ_{n_j} for which the integral in (0.0.3) is nonzero, in such a way that (0.0.3) be controlled by $C\prod\|u_j\|_{H^s}$ for s large enough. When $p=2$, which corresponds to a quadratic nonlinearity, and when the manifold $M=\mathbb{T}^d$, one can get a lower bound for $|F_m|$ by a negative power of the smallest of the three eigenvalues $\lambda_{n_1},\lambda_{n_2},\lambda_{n_3}$, whatever the value of $m>0$. This very special property is the key of the results obtained in [8] for quadratic nonlinearities on the torus. For higher order nonlinearities and for a general manifold, the only lower bounds one is able to get, when say p is an odd integer, hold true only for almost every m, and are of type

$$|F_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| \geq c(1+\lambda_{n_0}+\cdots+\lambda_{n_{p+1}})^{-N_0} \tag{0.0.5}$$

with a large enough N_0. Such estimates are useless when plugged in (0.0.3), as they make loose N_0 derivatives.

The situation is better when the manifold M is the sphere. In this case, using that the eigenvalues of $\sqrt{-\Delta}$ on $\mathbb{S}^d$ are the integers, up to a small perturbation, one can get, instead of (0.0.5), that for almost every $m > 0$, there are $c > 0, N_0 \in \mathbb{N}$ with

$$|F_m(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})| \geq c(1 + \text{ third largest among } (\lambda_{n_0}, \dots, \lambda_{n_{p+1}}))^{-N_0} \tag{0.0.6}$$

for any $n_0, \dots, n_{p+1}$ (still assuming for simplification that p is odd). In other words, the loss in (0.0.3) is given by a large power of a *small* frequency, which allows us to estimate, for $s \gg N_0$, (0.0.3) by $C \prod_j \|u_j\|_{H^s}$. Inequality (0.0.6) can be proved essentially because the set

$$\{\lambda_{n_i} - \lambda_{n_j}; n_i, n_j \in \mathbb{N}\} \tag{0.0.7}$$

is close to a discrete subset of $\mathbb{R}$. Such a property for the spectrum of a compact manifold holds true only in very special cases (see the paper of Guillemin [10] for more on this issue). For generic compact manifolds, (0.0.7) is actually dense in $\mathbb{R}$. This is in particular the case for the torus of dimension $d \geq 2$. Our main task will be to prove that in this case, in spite of the fact that an inequality as strong as (0.0.6) is not true, we may prove a weaker lower bound, using that we can use harmonic analysis on $\mathbb{T}^d$. We shall show that if $A > 1$ is given, for almost every m, there are $c > 0, N_0 \in \mathbb{N}$ such that

$$\begin{aligned} |F_m(|n_0|, \dots, |n_{p+1}|)| \geq c(1 + |n_0| + |n_{p+1}|)^{-d} \log(e + |n_0| + |n_{p+1}|)^{-A} \\ \times (1 + |n_0 - n_{p+1}|)^{-N_0}(1 + |n_1| + \cdots + |n_p|)^{-N_0} \end{aligned} \tag{0.0.8}$$

for any $n_0, \dots, n_{p+1} \in \mathbb{Z}^d$ with $|n_0|, |n_{p+1}| \gg |n_1| + \cdots + |n_p|$ (still assuming for simplification that p is odd). Comparing with (0.0.6), we see that division by F_m will not just make loose a power of low frequencies $|n_1|, \dots, |n_p|$. We shall also have a loss of d derivatives acting on high frequencies. To recover this, we shall use that equation (0.0.1) is weakly semi-linear (solving the linear equation makes gain one derivative, while the nonlinearity involves no derivative of v) and Hamiltonian. This last property allows one to gain one more derivative through commutators in energy inequalities. Consequently, the expressions to study are of form (0.0.3) but with the exponent $2s$ replaced by $2s - 2$. In the case $d = 2$ for instance, this shows that we may recover the loss of derivatives displayed by (0.0.8), up to a logarithm. In other words expressions of type (0.0.3) may be controlled by $C \prod \|u_j\|_{H^s}$, up to a logarithmic loss which may be transfered on a loss of type $|\log \epsilon|^A$ through partition of frequencies between zones $|n_0| + |n_{p+1}| < \epsilon^{-k}$ and $|n_0| + |n_{p+1}| \geq \epsilon^{-k}$ for some $k > 0$.

Let us give some hints on the way we prove (0.0.8). This inequality follows from the estimate of the measure of sets of form

$$\{m \in I; |F_m(|n_0|, \dots, |n_{p+1}|)| < r\} \tag{0.0.9}$$

where $I \subset]0, +\infty[$ is a compact interval and r is the right hand side of (0.0.8). We show, using tools of subanalytic geometry, that I may be written for any fixed $n_0, \dots, n_{p+1}$ as the union of a uniform number of intervals over which $|\partial F_m / \partial m|$ is bounded from below by a large negative power of small frequencies $(1 + |n_1| + \cdots + |n_p|)^{-N_1}$, and of a remaining set.

On each of these intervals, taking F_m as a coordinate, we estimate the measure of (0.0.9) by $Cr(1+|n_1|+\cdots+|n_p|)^{N_1}$. When r is given by the right hand side of (0.0.8), the sum of these quantities in $n_0,\dots,n_{p+1}$ is bounded by a small constant. The remaining set, corresponding to those m for which $|\partial F_m/\partial m| = O((1+|n_1|+\cdots+|n_p|)^{-N_1})$ may be also shown to be of small measure. This show that (0.0.8) holds true for all $n_0,\dots,n_{p+1}$ when m is outside a subset of small measure in I.

The above geometric properties will be obtained in subsection 2.1 of the paper, and used to give the proof of the main theorem of long time existence in subsection 2.2. This theorem is stated in subsection 1.1, which is followed by two subsections devoted to the reduction of the equation we study to a simpler form through paralinearization.

1 The semi-linear Klein-Gordon equation

1.1 Statement of the main theorem

Let d be an integer, $d \geq 2$, and set $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ for the standard torus. Denote by $\Box = \partial_t^2 - \Delta$ the d'Alembert operator on $\mathbb{R}\times\mathbb{T}^d$. Let $F : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$, $(x,v) \to F(x,v)$ be a real valued smooth function. We shall assume

$$\partial_v^j F(x,0) \equiv 0 \text{ for } j = 0,\dots,\kappa \tag{1.1.1}$$

for some $\kappa \in \mathbb{N}^*$. Let $m \in]0,+\infty[$. We consider the solution v of the Klein-Gordon equation

$$\begin{aligned} (\Box + m^2)v &= F(x,v) \\ v|_{t=0} &= \epsilon v_0 \\ \partial_t v|_{t=0} &= \epsilon v_1 \end{aligned} \tag{1.1.2}$$

where $v_0 \in H^{s+1}(\mathbb{T}^d,\mathbb{R})$, $v_1 \in H^s(\mathbb{T}^d,\mathbb{R})$, and $\epsilon > 0$ is small enough. By local existence theory, one knows that if s is large enough and $\epsilon \in]0,1[$, equation (1.1.2) admits for any (v_0,v_1) in the unit ball of $H^{s+1}\times H^s$ a unique smooth solution defined on the interval $|t| \leq c\epsilon^{-\kappa}$, for some uniform positive constant c. Moreover, $\|v(t,\cdot)\|_{H^{s+1}} + \|\partial_t v(t,\cdot)\|_{H^s}$ may be controlled by $C\epsilon$, for another uniform constant $C > 0$, on the interval of existence. The goal of this paper is to show that under convenient assumptions, one may extend such a solution and such an upper bound to an interval of length (almost) equal to $\epsilon^{-\kappa\left(1+\frac{2}{d}\right)}$. Let us state the main result.

Theorem 1.1.1 *There is a zero measure subset $\mathcal{N}$ of $]0,+\infty[$ and for every $m \in]0,+\infty[-\mathcal{N}$ and any $A > 1$, there are $s_0 > 0, c > 0, \epsilon_0 > 0$ such that for any $\epsilon \in]0,\epsilon_0[$, any (v_0,v_1) in the unit ball of $H^{s_0+1}(\mathbb{T}^d,\mathbb{R}) \times H^{s_0}(\mathbb{T}^d,\mathbb{R})$, equation (1.1.2) has a unique solution*

$$v \in C^0(]-T_\epsilon,T_\epsilon[,H^{s_0+1}) \cap C^1(]-T_\epsilon,T_\epsilon[,H^{s_0})$$

with $T_\epsilon \geq c\epsilon^{-\kappa\left(1+\frac{2}{d}\right)}|\log\epsilon|^{-A}$. Moreover, for any $s \geq s_0$, there are $\epsilon_s > 0, c_s > 0, C_s > 0$ such that when $\epsilon < \epsilon_s$ and (v_0,v_1) is in the unit ball of $H^{s+1}(\mathbb{T}^d,\mathbb{R}) \times H^s(\mathbb{T}^d,\mathbb{R})$, $\|v(t,\cdot)\|_{H^{s+1}} + \|\partial_t v(t,\cdot)\|_{H^s}$ is bounded by $C_s\epsilon$ for $|t| \leq c_s\epsilon^{-\kappa\left(1+\frac{2}{d}\right)}|\log\epsilon|^{-A}$.

Remarks • As already mentioned in the introduction, one can prove an almost global existence result (i.e. over an interval of length $c_N\epsilon^{-N}$ for any N) for equation (1.1.2) on $\mathbb{T}^1$. This has been done by Bourgain [5], Bambusi [1], Bambusi-Grébert [3]. Such an almost global theorem has been proved in higher dimensions as well by Bambusi, Delort, Grébert and Szeftel [2] for equation (1.1.2) on the *sphere* $\mathbb{S}^d$ (or more generally on a Zoll manifold).

• If one considers equation (1.1.2) on $\mathbb{S}^d$, with a nonlinearity of form $F(v, \partial_t v, \nabla_x v)$, with F homogeneous of even degree $\kappa + 1$, it has been proved by Delort and Szeftel [8, 9], that the solution exists over an interval of length $c\epsilon^{-2\kappa}$ (i.e. essentially the time of existence obtained in theorem 1.1.1 in dimension $d = 2$). We are unable in the case of the torus $\mathbb{T}^d$ ($d \geq 2$), to obtain a better existence interval than the one given by local existence theory, when the nonlinearities involve derivatives.

• The almost existence result of dimension 1 is obtained by an iterative method, allowing one to construct successive normal forms for the equation. We cannot hope for such a method to work on $\mathbb{T}^d(d \geq 2)$, since our first reduction will make us loose derivatives (because of the bad behaviour of the eigenvalues of $-\Delta$ on $\mathbb{T}^d$). This loss will be recovered because the right hand side of the equation contains no derivative of v. But the remainders which will be generated will not enjoy a similar structure, preventing us to iterate the argument.

1.2 Paradifferential operators and remainders

For $n \in \mathbb{Z}^d$ we set

$$\varphi_n(x) = \frac{1}{(2\pi)^{d/2}} e^{inx} \tag{1.2.1}$$

so that $(\varphi_n)_{n\in\mathbb{Z}^d}$ is an Hilbertian basis of $L^2(\mathbb{T}^d, \mathbb{C})$. For $u \in L^2(\mathbb{T}^d, \mathbb{C})$ we denote by $\Pi_n u$ the orthogonal projection of u on the span of φ_n and by $\hat{u}(n) = \langle u, \varphi_n \rangle$ so that

$$\Pi_n u = \hat{u}(n)\varphi_n(x). \tag{1.2.2}$$

Let us define the following class of operators:

Definition 1.2.1 *Let $p \in \mathbb{N}, \mu \in \mathbb{R}, \nu \in \mathbb{R}_+, \delta \in]0, 1[$. We denote by $\Sigma^{\mu,\nu}_{p,\delta}$ the space of maps*

$$\begin{aligned} (c, u_1, \ldots, u_p, \lambda) &\to a(c, u_1, \ldots, u_p, \lambda) \\ C^\infty(\mathbb{T}^d, \mathbb{C})^{p+1} \times \mathbb{R}^d &\to C^\infty(\mathbb{T}^d, \mathbb{C}) \end{aligned} \tag{1.2.3}$$

satisfying the following conditions:

(i) The map (1.2.3) is $\mathbb{C}$ $(p + 1)$-linear in $(c, u_1, \ldots, u_p)$ and smooth in λ.

(ii)$_\delta$ For any $n_0, \ldots, n_p \in \mathbb{Z}^d$, $\lambda \in \mathbb{R}$ such that $\max(|n_0|, \ldots, |n_p|) > \delta|\lambda|$, *one has*

$$a(\varphi_{n_0}, \ldots, \varphi_{n_p}, \lambda) \equiv 0. \tag{1.2.4}$$

Moreover, when $n_{p+1} \neq n_0 + \cdots + n_p$

$$\langle a(\varphi_{n_0}, \ldots, \varphi_{n_p}, \lambda), \varphi_{n_{p+1}} \rangle \equiv 0. \tag{1.2.5}$$

(iii) For any $\alpha, \beta \in \mathbb{N}^d$*, there is* $C > 0$ *such that for any* $n_0, \ldots, n_p \in \mathbb{Z}^d$*, any* $\lambda \in \mathbb{R}^d$

$$\|\partial_x^\alpha \partial_\lambda^\beta a(\varphi_{n_0}, \ldots, \varphi_{n_p}, \lambda)\|_{L^\infty(\mathbb{T}^d)} \leq C\langle\lambda\rangle^{\mu-|\beta|}(1 + |n_0| + \cdots + |n_p|)^{\nu+|\alpha|}. \tag{1.2.6}$$

Remark Inequality (1.2.6) shows that the map (1.2.3) may be extended to $H^s(\mathbb{T}^d)^{p+1} \times \mathbb{R}^d$ for s large enough.

An example of a symbol satisfying the conditions of definition 1.2.1 may be obtained as follows. Let $\lambda \to b(\lambda)$ be a symbol of order μ on $\mathbb{R}^d$ (in the usual sense). Let $A(X_0, \ldots, X_p)$ be a $p+1$ linear form on $(\mathbb{C}^d)^{p+1}$, and let $\chi \in C_0^\infty(\mathbb{R})$. Define if $\gamma_0, \ldots, \gamma_p \in \mathbb{R}$

$$a(c, u_1, \ldots, u_p, \lambda) = \sum_{n_0, \ldots, n_p} \chi\Big(\frac{\max(|n_0|, \ldots, |n_p|)}{\sqrt{1+\lambda^2}}\Big) A(\Lambda_m^{\gamma_0}\Pi_{n_0}c, \Lambda_m^{\gamma_1}\Pi_{n_1}u_1, \ldots, \Lambda_m^{\gamma_p}\Pi_{n_p}u_p)b(\lambda) \tag{1.2.7}$$

where $\Lambda_m = \sqrt{-\Delta + m^2}$. Then we get an element of $\Sigma_{p,\delta}^{\mu,\nu}$ if $\operatorname{Supp}\chi$ is small enough and ν is large enough. Actually, all symbols we shall have to deal with will be of form (1.2.7).

We shall use also classes of multilinear operators, for which we shall be interested only in less precise properties.

Definition 1.2.2 *Let* $p \in \mathbb{N}, \mu \in \mathbb{R}, \nu \in \mathbb{R}_+, \delta \in]0, 1[$. *We denote by* $\mathcal{M}_{p+1,\delta}^{\mu,\nu}$ *the space of all* $\mathbb{C}$ $(p+1)$*-linear maps* $(u_1, \ldots, u_{p+1}) \to M(u_1, \ldots, u_{p+1})$*, defined on* $C^\infty(\mathbb{T}^d)^{p+1}$*, with values in* $L^2(\mathbb{T}^d)$*, such that*

(i) For any $n_0, \ldots, n_{p+1} \in \mathbb{Z}^d$*, any* $u_1, \ldots, u_{p+1} \in C^\infty(\mathbb{T}^d)$

$$\Pi_{n_0}[M(\Pi_{n_1}u_1, \ldots, \Pi_{n_{p+1}}u_{p+1})] \equiv 0 \tag{1.2.8}$$

if $|n_0 - n_{p+1}| > \delta(|n_0| + |n_{p+1}|)$ *or* $|n'| \stackrel{\text{def}}{=} \max(|n_1|, \ldots, |n_p|) > \delta(|n_0| + |n_{p+1}|)$.

(ii) For any $N \in \mathbb{N}$*, there is* $C_N > 0$ *such that for any* $u_1, \ldots, u_{p+1} \in C^\infty(\mathbb{T}^d)$*, any* $n_0, \ldots, n_{p+1}$ *in* $\mathbb{Z}^d$*,*

$$\|\Pi_{n_0}[M(\Pi_{n_1}u_1, \ldots, \Pi_{n_{p+1}}u_{p+1})]\|_{L^2} \leq C_N(1+|n_0|+|n_{p+1}|)^\mu \frac{(1+|n'|)^{\nu+N}}{(|n_0 - n_{p+1}| + |n'| + 1)^N} \prod_{j=1}^{p+1} \|u_j\|_L^2. \tag{1.2.9}$$

The best constant C_N *in the preceding inequality will be denoted* $\|M\|_{\mathcal{M}_{p+1,\delta}^{\mu,\nu}(N)}$.

We may extend the action of operators in $\mathcal{M}_{p+1,\delta}^{\mu,\nu}$ to Sobolev spaces. Actually, it follows from the above conditions:

Lemma 1.2.3 *Let $p \in \mathbb{N}, \mu \in \mathbb{R}, \nu \in \mathbb{R}_+, \delta \in]0,1[$. There is $s_0 \in \mathbb{R}_+$ such that for any $s \geq s_0$, any $M \in \mathcal{M}^{\mu,\nu}_{p+1,\delta}$ may be extended as a continuous map from $H^{s_0}(\mathbb{T}^d) \times \cdots \times H^{s_0}(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$ to $H^{s-\mu}(\mathbb{T}^d)$, with norm controlled by $C\|M\|_{\mathcal{M}^{\mu,\nu}_{p+1,\delta}(d+1)}$.*

Let us define now from the class of multilinear symbols of definition 1.2.1 another family of symbols.

Definition 1.2.4 *(i) Let $p \in \mathbb{N}, \mu \in \mathbb{R}, \nu \in \mathbb{R}_+, \delta \in]0,1[$. We denote by $S^{\mu,\nu}_{p,\delta}$ the space of maps $(u,\lambda) \to a(u,\bar{u},\lambda)$ defined on $C^\infty(\mathbb{T}^d,\mathbb{C}) \times \mathbb{R}^d$, with values in $C^\infty(\mathbb{T}^d,\mathbb{C})$, such that there is a family a_ℓ of elements of $\Sigma^{\mu,\nu}_{p,\delta}$ $\ell = 0,\dots,p$, and a family of functions $c_\ell \in C^\infty(\mathbb{T}^d,\mathbb{C})$, such that*

$$a(u,\bar{u};\lambda) = \sum_{\ell=0}^{p} a_\ell(c_\ell, \underbrace{\bar{u},\dots,\bar{u}}_{\ell}, \underbrace{u,\dots,u}_{p-\ell}; \lambda). \tag{1.2.10}$$

(ii) One denotes by $S^{\mu,\nu}_\delta = \bigoplus_{p\geq 0} S^{\mu,\nu}_{p,\delta}$.

We now quantize elements of the preceding class of symbols.

Definition 1.2.5 *Let $a \in S^{\mu,\nu}_\delta$. For $u, w \in C^\infty(\mathbb{T}^d,\mathbb{C})$ we define*

$$\mathrm{Op}(a(u,\bar{u};\cdot))w = \frac{1}{(2\pi)^{d/2}} \sum_{n\in\mathbb{Z}^d} e^{inx} a(u(x),\bar{u}(x);n)\hat{w}(n). \tag{1.2.11}$$

Let us remark that the above operators may be written in terms of the multilinear maps of definition 1.2.2. Actually, let us define using notation (1.2.10)

$$M_\ell(u_1,\dots,u_{p+1}) = \frac{1}{(2\pi)^{d/2}} \sum_{n\in\mathbb{Z}^d} e^{inx} a_\ell(c_\ell, u_1,\dots,u_p; n)\hat{u}_{p+1}(n).$$

We have

$$\Pi_{n_0} M_\ell(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}) = \langle \varphi_{n_{p+1}} a_\ell(c_\ell, \varphi_{n_1},\dots,\varphi_{n_p}; n_{p+1}), \varphi_{n_0}\rangle \varphi_{n_0} \hat{u}_1(n_1)\cdots\hat{u}_{p+1}(n_{p+1}). \tag{1.2.12}$$

The bracket may be written

$$\frac{1}{(2\pi)^{d/2}} \sum_{k} \langle a_\ell(\varphi_k, \varphi_{n_1},\dots,\varphi_{n_p}; n_{p+1}), \varphi_{n_0-n_{p+1}}\rangle \hat{c}_\ell(k). \tag{1.2.13}$$

Consequently, by condition (1.2.5) we must have $n_0 - n_{p+1} = k + n_1 + \cdots + n_p$, whence by (1.2.6) and since c_ℓ is C^∞, an estimate of (1.2.13) by

$$C_N \langle n_{p+1}\rangle^\mu (1+|n_1|+\cdots+|n_p|+|n_0-n_1-\cdots-n_{p+1}|)^\nu (1+|n_0-n_1-\cdots-n_{p+1}|)^{-N} \tag{1.2.14}$$

for any N. Moreover, (1.2.4) implies that we must have

$$\max(|n_0-n_1-\cdots-n_{p+1}|, |n_1|,\dots,|n_p|) < \delta|n_{p+1}|.$$

If $\delta > 0$ is small enough, this implies that condition (1.2.8) of definition 1.2.2 is satisfied by (1.2.12) for a new value of δ. Moreover (1.2.14) implies (1.2.9). We have proved:

Lemma 1.2.6 *Let $p \in \mathbb{N}, \mu \in \mathbb{R}, \nu \in \mathbb{R}_+, \delta > 0$ small enough. There is $\delta' \in]0,1[$ and for any $a \in S^{\mu,\nu}_{p,\delta}$, there are elements $M_\ell \in \mathcal{M}^{\mu,\nu}_{p+1,\delta'}$ $\ell = 0, \dots, p$ such that*

$$\text{Op}(a(u,\bar{u};\cdot))w = \sum_{\ell=0}^{p} M_\ell(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p-\ell},w) \tag{1.2.15}$$

for any $u, w \in C^\infty(\mathbb{T}^d, \mathbb{C})$.

Remark If we use lemma 1.2.3, we see that there is $s_0 > 0$ such that for any $a \in S^{\mu,\nu}_{p,\delta}$, the map $(u,w) \to \text{Op}(a(u,\bar{u};\cdot))w$ extends as a continuous map from $H^{s_0}(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$ to $H^{s-\mu}(\mathbb{T}^d)$ for any $s \geq s_0$.

We shall now establish a result of symbolic calculus.

Proposition 1.2.7 *Let $p \in \mathbb{N}, \mu \in \mathbb{R}, \nu \in \mathbb{R}_+, \delta > 0$ small enough. There are $\nu' > 0, \delta' \in]0,1[$ such that for any real valued $a \in S^{\mu,\nu}_{p,\delta}$, any $s \in \mathbb{R}_+$, one may find operators $M_\ell \in \mathcal{M}^{2s+\mu-1,\nu'}_{p+1,\delta'}$ $\ell = 0, \dots, p$ such that*

$$\langle(\Lambda_m^{2s}\text{Op}(a(u,\bar{u};\cdot)) - \text{Op}(a(u,\bar{u};\cdot))^*\Lambda_m^{2s})u, u\rangle = \sum_{\ell=0}^{p} \langle M_\ell(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}), u\rangle. \tag{1.2.16}$$

Moreover, if one assumes that $a(u,\bar{u};n) \equiv a(u,\bar{u};n')$ when $|n| = |n'|$,

$$\sum_{\ell=0}^{p} \Pi_{n_0} M_\ell(\bar{u},\dots,\bar{u},u,\dots,u,\Pi_{n_{p+1}}u) \equiv 0 \tag{1.2.17}$$

holds true for any $u \in C^\infty(\mathbb{T}^d, \mathbb{C})$, any $n_0, n_{p+1} \in \mathbb{Z}^d$ with $|n_0| = |n_{p+1}|$.

Proof: Let us denote by $A(x,n) = a(u(x),\bar{u}(x);n)$ and by $\hat{A}(k,n) = \langle A(\cdot,n), \varphi_k\rangle$. Then we may write for any $u, v \in C^\infty(\mathbb{T}^d, \mathbb{C})$

$$\begin{aligned}\langle \text{Op}(a(u,\bar{u};\cdot))u, v\rangle &= \frac{1}{(2\pi)^{d/2}} \sum_n \int_{\mathbb{T}^d} e^{inx} A(x,n)\hat{u}(n)\overline{v(x)}\,dx \\ &= \frac{1}{(2\pi)^{d/2}} \sum_n \sum_k \hat{A}(k-n,n)\hat{u}(n)\overline{\hat{v}(k)}.\end{aligned}$$

By an immediate computation, we get also

$$\langle \text{Op}(a(u,\bar{u};\cdot))^* u, v\rangle = \frac{1}{(2\pi)^{d/2}} \sum_n \sum_k \hat{\bar{A}}(k-n,k)\hat{u}(n)\overline{\hat{v}(k)}.$$

Denote by $\lambda_m(n) = \sqrt{m^2 + |n|^2}$. Using that a is real valued, we may write

$$\begin{aligned}&\langle(\Lambda_m^{2s}\text{Op}(a(u,\bar{u};\cdot)) - \text{Op}(a(u,\bar{u};\cdot))^*\Lambda_m^{2s})u, v\rangle \\ &= \frac{1}{(2\pi)^{d/2}} \sum_n \sum_k [\lambda_m(k)^{2s}\hat{A}(k-n,n) - \lambda_m(n)^{2s}\hat{A}(k-n,k)]\hat{u}(n)\overline{\hat{v}(k)} \\ &= \langle \hat{M}, \hat{v}\rangle\end{aligned} \tag{1.2.18}$$

where $\hat{M}(k)$ is defined by

$$
\begin{aligned}
\hat{M}(k) &= \frac{1}{(2\pi)^{d/2}} \sum_n b(k,n)\hat{u}(n) \\
b(k,n) &= \lambda_m(k)^{2s}\hat{A}(k-n,n) - \lambda_m(n)^{2s}\hat{A}(k-n,k).
\end{aligned} \tag{1.2.19}
$$

Let us decompose a as in (1.2.10) and define the scalar quantity
(1.2.20)

$$
b_\ell(u_1,\dots,u_p;k,n) = \langle \lambda_m(k)^{2s} a_\ell(c_\ell,u_1,\dots,u_p;n) - \lambda_m(n)^{2s} a_\ell(c_\ell,u_1,\dots,u_p;k), \varphi_{k-n}\rangle.
$$

Set

$$
M_\ell(u_1,\dots,u_p) = \frac{1}{(2\pi)^d} \sum_n \sum_k e^{ikx} b_\ell(u_1,\dots,u_p;k,n)\hat{u}_{p+1}(n). \tag{1.2.21}
$$

Remark that (1.2.20) and (1.2.19) and the definition of A imply that (1.2.17) holds true. Moreover

$$
\begin{aligned}
\sum_{\ell=0}^{p} \langle M_\ell(\underbrace{\bar{u},\dots,\bar{u}}_{\ell}, \underbrace{u,\dots,u}_{p+1-\ell}), u\rangle &= \frac{1}{(2\pi)^{d/2}} \sum_{\ell=0}^{p} \sum_n \sum_k b_\ell(\bar{u},\dots,\bar{u},u\dots,u;k,n)\hat{u}(n)\overline{\hat{u}(k)} \\
&= \langle \hat{M}, \hat{u}\rangle,
\end{aligned}
$$

which because of (1.2.18) implies (1.2.16). Let us check that each M_ℓ belongs to $\mathcal{M}^{2s+\mu-1,\nu'}_{p+1,\delta'}$. Let us compute first $b_\ell(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p;k,n)$ from expression (1.2.20). By condition (ii) of definition 1.2.1, if this quantity is nonzero, we must have $\max(|n_1|,\dots,|n_p|) < \delta(|n|+|k|)$, and we may assume that c_ℓ has nonzero modes only for frequencies smaller that $\delta(|n|+|k|)$. Consequently, using (1.2.5), we see that $b_\ell(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p;k,n)$ is supported for

$$
|n-k| \le C\delta(|n|+|k|) \text{ and } |n'| = \max(|n_1|,\dots,|n_p|) \le \delta(|n|+|k|). \tag{1.2.22}
$$

We shall assume that $\delta > 0$ is small enough so that $C\delta < 1$. Since λ_m is a symbol of order 1, and a_ℓ satisfies (1.2.6), it follows from (1.2.20) and the fact that $\hat{c}_\ell$ is rapidly decaying that

$$
|b_\ell(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p;k,n)| \le C(1+|k|+|n|)^{2s+\mu-1}(1+|k-n|)(1+|n'|)^\nu \prod_1^p |\hat{u}_j(n_j)|.
$$

If in (1.2.20) we write $\varphi_{k-n} = [\lambda_m(k-n)]^{-2}(-\Delta+m^2)\varphi_{k-n}$ and perform integrations by parts, we get in the same way an upper bound in terms of

$$
C(1+|k|+|n|)^{2s+\mu-1}(1+|k-n|)^{1-N}(1+|n'|)^{\nu+N} \prod_1^p |\hat{u}_j(n_j)|.
$$

It follows from these inequalities that

$$
\begin{aligned}
&\|\Pi_{n_0}M_\ell(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})\|_{L^2} \le \\
&\qquad C(1+|n_0|+|n_{p+1}|)^{2s+\mu-1} \frac{(1+|n'|)^{\nu+1+N}}{(|n_0-n_{p+1}|+|n'|+1)^N} \prod_1^{p+1} \|u_j\|_{L^2},
\end{aligned}
$$

which is the wanted estimate of type (1.2.9). Property (1.2.8) follows from (1.2.22). This concludes the proof of the proposition. $\square$

We shall have to use also classes of remainder operators. If $n_1, \dots, n_{p+1} \in \mathbb{Z}^d$ and if $i_0 \in \{1, \dots, p+1\}$ is such that $|n_{i_0}| = \max(|n_1|, \dots, |n_{p+1}|)$, we denote

$$\max{}_2(|n_1|, \dots, |n_{p+1}|) = \max\{|n_j|; 1 \leq j \leq p+1, j \neq j_0\} + 1. \tag{1.2.23}$$

Definition 1.2.8 *Let $p \in \mathbb{N}, \mu \in \mathbb{R}, \nu \in \mathbb{R}_+$. We denote by $\mathcal{R}^{\mu,\nu}_{p+1}$ the space of $\mathbb{C}$ $(p+1)$-linear maps from $\mathbb{C}^\infty(\mathbb{T}^d, \mathbb{C})^{p+1}$ to $L^2(\mathbb{T}^d, \mathbb{C})$, $(u_1, \dots, u_{p+1}) \to R(u_1, \dots, u_{p+1})$ such that for any $N \in \mathbb{N}$, there is $C > 0$ such that for any $n_0, \dots, n_{p+1} \in \mathbb{Z}^d$, any $u_1, \dots, u_{p+1} \in C^\infty(\mathbb{T}^d, \mathbb{C})$*

$$\|\Pi_{n_0} R(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1})\|_{L^2} \leq C(1+|n_0|)^\mu \frac{\max_2(|n_1|, \dots, |n_{p+1}|)^{\nu+N}}{(1+|n_0|+\cdots+|n_{p+1}|)^N} \prod_{j=1}^{p+1} \|u_j\|_{L^2}. \tag{1.2.24}$$

We have

Lemma 1.2.9 *Let $p \in \mathbb{N}, \nu \in \mathbb{R}_+$ be given. There is $s_0 \in \mathbb{R}_+$ such that for any $s > s_0$, any $\mu \in \mathbb{R}$, any $R \in \mathcal{R}^{\mu,\nu}_{p+1}$, $(u_1, \dots, u_{p+1}) \to R(u_1, \dots, u_{p+1})$ extends as a bounded map from $H^s(\mathbb{T}^d) \times \cdots \times H^s(\mathbb{T}^d)$ to $H^{2s-\mu-\nu-2(d+1)}(\mathbb{T}^d)$. Moreover, one can estimate*

$$\|R(u_1, \dots, u_{p+1})\|_{H^{2s-\mu-\nu-2(d+1)}} \leq C \sum_{1 \leq j_1 < j_2 \leq p+1} \Big(\prod_{k \neq j_1, j_2} \|u_k\|_{H^{s_0}}\Big) \|u_{j_1}\|_{H^s} \|u_{j_2}\|_{H^s}. \tag{1.2.25}$$

Proof: We may assume that $\mu = 0$. We bound $\|\Pi_{n_0} R(u_1, \dots, u_{p+1})\|_{L^2}$ decomposing u_j as $\sum_{n_j} \Pi_{n_j} u_j$ and using (1.2.24). By symmetry we limit ourselves to summation over $|n_1| \leq \cdots \leq |n_p| \leq |n_{p+1}|$ so that we have to bound

$$\begin{aligned}\sum_{|n_1| \leq \cdots \leq |n_p| \leq |n_{p+1}|} \frac{(1+|n_p|)^{\nu+N}}{(1+|n_0|+\cdots+|n_{p+1}|)^N} \prod_1^{p-1} (1+|n_j|)^{-s_0} (1+|n_p|)^{-s} (1+|n_{p+1}|)^{-s} c_{n_{p+1}} \\ \times \prod_1^{p-1} \|u_j\|_{H^{s_0}} \|u_p\|_{H^s} \|u_{p+1}\|_{H^s}\end{aligned} \tag{1.2.26}$$

for a ℓ^2 sequence $(c_{n_{p+1}})_{n_{p+1}}$. When we sum for $|n_{p+1}| \geq \frac{1}{2}|n_0|$ we take $N = 2s$. We get for the general term of (1.2.26) the upper bound

$$\begin{aligned}&C \prod_1^{p-1} (1+|n_j|)^{-s_0} (1+|n_p|)^{\nu+s} (1+|n_{p+1}|)^{-3s+d+1} (1+|n_0-n_{p+1}|)^{-d-1} c_{n_{p+1}} \\ &\quad \leq C \prod_1^{p-1} (1+|n_j|)^{-s_0} (1+|n_p|)^{-d-1} (1+|n_0-n_{p+1}|)^{-d-1} c_{n_{p+1}} (1+|n_0|)^{-2s+\nu+2(d+1)}\end{aligned}$$

using that on the summation $|n_{p+1}| \geq \frac{1}{2}|n_0|$ and $|n_p| \leq |n_{p+1}|$, and taking s_0 large enough so that $2s \geq \nu + 2(d+1)$. If we sum for $|n_{p+1}| < \frac{1}{2}|n_0|$, we take $N = 2s - \nu - d - 1$. We get for the general term of (1.2.26) the upper bound

$$(1+|n_0|)^{-2s+\nu+2(d+1)}(1+|n_0-n_{p+1}|)^{-d-1}\prod_1^{p-1}(1+|n_j|)^{-s_0}(1+|n_p|)^{s-d-1}(1+|n_{p+1}|)^{-s}c_{n_{p+1}}$$

$$\leq C(1+|n_0|)^{-2s+\nu+2(d+1)}(1+|n_0-n_{p+1}|)^{-d-1}\prod_1^{p-1}(1+|n_j|)^{-s_0}(1+|n_p|)^{-d-1}c_{n_{p+1}}.$$

We get in both cases for the $n_1, \ldots, n_{p+1}$ sum an upper bound of type $(1+|n_0|)^{-2s+\nu+2(d+1)}c'_{n_0}$, for a new ℓ^2 sequence $(c'_{n_0})_{n_0}$. $\square$

To conclude this subsection, let us introduce another class of operators.

Definition 1.2.10 *Let $p \in \mathbb{N}, \mu \in \mathbb{R}, \nu \in \mathbb{R}_+$. We denote by $\widetilde{\mathcal{R}}^{\mu,\nu}_{p+1}$ the space of maps $u \to R(u)$ defined on $C^\infty(\mathbb{T}^d, \mathbb{C})$ with values in $L^2(\mathbb{T}^d, \mathbb{C})$ such that there is a family of elements $R_\ell \in \mathcal{R}^{\mu,\nu}_{p+1}$, $\ell = 0, \ldots, p+1$ satisfying*

$$R(u) = \sum_{\ell=0}^{p+1} R_\ell(\underbrace{\bar{u}, \ldots, \bar{u}}_{\ell}, \underbrace{u, \ldots, u}_{p+1-\ell}). \tag{1.2.27}$$

When p is odd, we set $\widetilde{\mathcal{R}}'^{\mu,\nu}_{p+1} = \widetilde{\mathcal{R}}^{\mu,\nu}_{p+1}$. When p is even, we denote by $\widetilde{\mathcal{R}}'^{\mu,\nu}_{p+1}$ the subspace of $\widetilde{\mathcal{R}}^{\mu,\nu}_{p+1}$ made of those R such that in decomposition (1.2.27) the term indexed by $\ell = p/2$ satisfies

$$\operatorname{Im} \sum_{n_0,\ldots,n_{p+1}}{}' \langle \Pi_{n_0} R_\ell(\Pi_{n_1}\bar{u}, \ldots, \Pi_{n_\ell}\bar{u}, \Pi_{n_{\ell+1}}u, \ldots, \Pi_{n_{p+1}}u), u\rangle \equiv 0 \tag{1.2.28}$$

for any $u \in C^\infty(\mathbb{T}^d, \mathbb{C})$, where $\sum'$ stands for the sum extended over all $(n_0, \ldots, n_{p+1}) \in (\mathbb{Z}^d)^{p+2}$ such that there is a bijection $\sigma : \{0, \ldots, \ell\} \to \{\ell+1, \ldots, p+1\}$ with $|n_{\sigma(j)}| = |n_j|$ for $j = 0, \ldots, \ell$.

1.3 Paralinearization of the equation

Our goal in this subsection is to write equation (1.1.2) using a paradifferential expression for the nonlinearity. We shall make a change of unknown, writing with $\Lambda_m = \sqrt{-\Delta + m^2}$,

$$u = (D_t + \Lambda_m)v, \; v = \frac{1}{2}\Lambda_m^{-1}(u + \bar{u}) \tag{1.3.1}$$

so that (1.1.2) may be written

$$\begin{aligned} (D_t - \Lambda_m)u &= -F(x, \frac{1}{2}\Lambda_m^{-1}(u+\bar{u})) \\ u|_{t=0} &= \epsilon u_0 \end{aligned} \tag{1.3.2}$$

with $u_0 = -iv_1 + \Lambda_m v_0 \in H^s(\mathbb{T}^d, \mathbb{C})$. The main result of this subsection is the following one:

Proposition 1.3.1 *Let $\delta \in]0,1[$ be given. There is $\nu \in \mathbb{R}_+$, there are for $p = \kappa, \dots, 2\kappa - 1$ real valued symbols $a^p \in S^{-1,\nu}_{p,\delta}$ and remainder operators $R^p \in \widetilde{\mathcal{R}}'^{0,\nu}_{p+1}$, there is a map $u \to S(u)$ satisfying for any $s_0 > d/2$, any $s \geq s_0$, and any u in $H^s(\mathbb{T}^d, \mathbb{C})$ belonging to the unit ball of $H^{s_0}(\mathbb{T}^d, \mathbb{C})$,*

$$\|S(u)\|_{H^s} \leq C_s \|u\|^{2\kappa}_{H^{s_0}} \|u\|_{H^s},$$

such that the first equation in (1.3.2) may be written

$$(D_t - \Lambda_m)u = \sum_{p=\kappa}^{2\kappa-1} \mathrm{Op}(a^p(u, \bar{u}; \cdot))(u + \bar{u}) + \sum_{p=\kappa}^{2\kappa-1} R^p(u) + S(u). \tag{1.3.3}$$

Moreover, one may assume that $a^p(u, \bar{u}; n) = a^p(u, \bar{u}; n')$ if $|n| = |n'|$.

Proof: We decompose

$$-F(x, v) = -\sum_{p=\kappa}^{2\kappa-1} \frac{(\partial_v^{p+1} F)(x, 0)}{(p+1)!} v^{p+1} + G(x, v) \tag{1.3.4}$$

where $G(x, v)$ vanishes at order $2\kappa + 1$ at $v = 0$. The contribution of G will be incorporated in the S term of (1.3.3). We have to treat each term in the right hand side of (1.3.4) i.e. quantities of type $c(x)v^{p+1}$ where c is smooth and real valued. We decompose

$$c(x)v^{p+1} = \sum_k \sum_{n_1} \cdots \sum_{n_{p+1}} (\Pi_k c)(\Pi_{n_1} v) \cdots (\Pi_{n_{p+1}} v). \tag{1.3.5}$$

We decompose (1.3.5) as a sum of terms for which $|k| + \max_2(|n_1|, \dots, |n_{p+1}|)$ is much smaller than $\max(|n_1|, \dots, |n_{p+1}|)$ and a remaining term: take $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero, $\operatorname{Supp} \chi$ small enough. Then (1.3.5) is the sum of

$$\sum_k \sum_{n_1} \cdots \sum_{n_{p+1}} \chi\left(\frac{|k| + |n'|}{\sqrt{1 + |n_{p+1}|^2}} \right) (\Pi_k c)(\Pi_{n_1} v) \cdots (\Pi_{n_{p+1}} v) \tag{1.3.6}$$

(where $|n'| = \max(|n_1|, \dots, |n_p|)$), of terms of the same type obtained through permutation of $n_1, \dots, n_{p+1}$, and of

$$\sum_k \sum_{n_1} \cdots \sum_{n_{p+1}} A(|k|, |n_1|, \dots, |n_{p+1}|)(\Pi_k c)(\Pi_{n_1} v) \cdots (\Pi_{n_{p+1}} v) \tag{1.3.7}$$

where A stands for a real valued bounded function, supported inside the domain

$$|k| + \max_2(|n_1|, \dots, |n_{p+1}|) \geq c \max(|n_1|, \dots, |n_{p+1}|) \tag{1.3.8}$$

for some $c > 0$, and invariant under permutations of $|n_1|, \dots, |n_{p+1}|$. Define

$$\begin{aligned} R^p_\ell(u_1, \dots, u_{p+1}) = \binom{p+1}{\ell} \frac{1}{2^{p+1}} \sum_k \sum_{n_1} \cdots \sum_{n_{p+1}} A(|k|, |n_1|, \dots, |n_{p+1}|) \\ \times (\Pi_k c)(\Pi_{n_1} \Lambda_m^{-1} u_1) \cdots (\Pi_{n_{p+1}} \Lambda_m^{-1} u_{p+1}) \end{aligned} \tag{1.3.9}$$

and

$$R^p(u) = \sum_{\ell=0}^{p+1} R_\ell^p(\underbrace{\bar{u}, \dots, \bar{u}}_{\ell}, u \dots, u). \tag{1.3.10}$$

Then (1.3.7) is given by $R^p(u)$ and so contributes to the second term in the right hand side of (1.3.3) if we show that $R^p \in \widetilde{\mathcal{R}}'^{0,\nu}_{p+1}$. Set $k_0 = n_0 - n_1 - \cdots - n_{p+1}$. Then

$$\begin{aligned} &\Pi_{n_0} R_\ell^p(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1}) = \\ &\binom{p+1}{\ell} \frac{1}{2^{p+1}} A(|k_0|, |n_1|, \dots, |n_{p+1}|) \Pi_{n_0}[(\Pi_{k_0} c)(\Pi_{n_1} \Lambda_m^{-1} u_1) \cdots (\Pi_{n_{p+1}} \Lambda_m^{-1} u_{p+1})]. \end{aligned} \tag{1.3.11}$$

Using Sobolev injection, we see that the L^2 norm of this quantity is bounded from above for any N by

$$C_N(1 + |k_0|)^{-N}(1 + \max_2(|n_1|, \dots, |n_{p+1}|))^\nu \prod_1^{p+1} \|u_j\|_{L^2}$$

for some ν depending only on p. If $|k_0| \geq \frac{c}{2} \max(|n_1|, \dots, |n_{p+1}|)$ we get an upper bound of form (1.2.24). If $|k_0| < \frac{c}{2} \max(|n_1|, \dots, |n_{p+1}|)$, it follows from (1.3.8) that

$$\max_2(|n_1|, \dots, |n_{p+1}|) \geq \frac{c}{2} \max(|n_1|, \dots, |n_{p+1}|)$$

and from the equality $n_0 = k_0 + n_1 + \cdots + n_{p+1}$ that $|n_0| \leq C \max(|n_1|, \dots, |n_{p+1}|)$. Consequently, estimate (1.2.24) for (1.3.11) is trivial, and $R_\ell^p \in \mathcal{R}^{0,\nu}_{p+1}$. When p is even and $\ell = p/2$, the left hand side of (1.2.28), with R_ℓ replaced by R_ℓ^p given by (1.3.11), equals

$$\begin{aligned} \operatorname{Im}\Bigg[&{\sum_{n_0,\dots,n_{p+1}}}' \sum_k A(|k|, |n_1|, \dots, |n_{p+1}|) \frac{1}{2^{p+1}} \binom{p+1}{\ell} \prod_{j=1}^{p+1} (1 + |n_j|^2)^{-1/2} \\ &\times \int_{\mathbb{T}^d} (\Pi_k c)(\Pi_{-n_0} \bar{u})(\Pi_{n_1} \bar{u}) \cdots (\Pi_{n_\ell} \bar{u})(\Pi_{n_{\ell+1}} u) \cdots (\Pi_{n_{p+1}} u)\, dx \Bigg]. \end{aligned} \tag{1.3.12}$$

Denote

$$\widetilde{\Pi}_\lambda = \frac{1}{\#\{n \in \mathbb{Z}^d; |n| = \lambda\}} \sum_{n; |n| = \lambda} \Pi_n.$$

Then we can in (1.3.12) replace the integral by the quantity

$$\int_{\mathbb{T}^d} (\widetilde{\Pi}_{|k|} c)(\widetilde{\Pi}_{|n_0|} \bar{u}) \cdots (\widetilde{\Pi}_{|n_\ell|} \bar{u})(\widetilde{\Pi}_{|n_{\ell+1}|} u) \cdots (\widetilde{\Pi}_{|n_{p+1}|} u)\, dx$$

which is real since c is real, and since $(n_0, \dots, n_{p+1})$ verify the condition defining the $\sum'$ sum in (1.2.28). Consequently (1.3.12) vanishes identically, which shows that $R^p \in \widetilde{\mathcal{R}}'^{0,\nu}_{p+1}$.

To finish the proof of the proposition, we are left with showing that (1.3.6) may be written as a contribution to the first term in the right hand side of (1.3.3). Define

$$a^p(u, \bar{u}; n_{p+1}) = \sum_k \sum_{n_1} \cdots \sum_{n_p} (\Pi_k c)(\Pi_{n_1} v) \cdots (\Pi_{n_p} v) \chi\left(\frac{|k| + |n'|}{\sqrt{1 + |n_{p+1}|^2}}\right) \frac{1}{2}(m^2 + |n_{p+1}|^2)^{-1/2}. \tag{1.3.13}$$

Since c and v are real valued, and since $\overline{\Pi_n v} = \Pi_{-n} v$, we see that a^p is real valued. Set for $\lambda \in \mathbb{R}^d$

$$b(c, u_1, \dots, u_p; \lambda) = \frac{1}{2^p} \sum_k \sum_{n_1} \cdots \sum_{n_p} (\Pi_k c)(\Pi_{n_1} \Lambda_m^{-1} u_1) \cdots (\Pi_{n_p} \Lambda_m^{-1} u_p) \times \chi\Big(\frac{|k| + |n'|}{\sqrt{1+\lambda^2}}\Big) \frac{1}{2}(m^2 + \lambda^2)^{-1/2}.$$

Then by the example following definition 1.2.1, $b \in \Sigma_{p,\delta}^{-1,\nu}$ for some $\nu > 0$, and for any given $\delta > 0$ if $\operatorname{Supp}\chi$ is taken small enough. Moreover

$$a^p(u, \bar{u}; \lambda) = \sum_{\ell=0}^{p} \binom{p}{\ell} b(c, \underbrace{\bar{u}, \dots, \bar{u}}_{\ell}, u, \dots, u; \lambda) \in S_{p,\delta}^{-1,\nu}$$

and by definition 1.2.5, (1.3.6) equals $\mathrm{Op}(a^p(u, \bar{u}; \cdot))(u + \bar{u})$ and so contributes to the first term in the right hand side of (1.3.3). Moreover, by (1.3.13), $a^p(u, \bar{u}; n_{p+1}) = a^p(u, \bar{u}; n'_{p+1})$ if $|n_{p+1}| = |n'_{p+1}|$. $\square$

2 Proof of the main theorem

2.1 Geometric bounds

Consider the function on $(\mathbb{R}^d)^{p+2}$ depending on the parameter $m \in]0, +\infty[$, defined for $\ell = 0, \dots, p+1$ by

$$F_m^\ell(\xi_0, \dots, \xi_{p+1}) = \sum_{j=0}^{\ell} \sqrt{m^2 + |\xi_j|^2} - \sum_{j=\ell+1}^{p+1} \sqrt{m^2 + |\xi_j|^2}. \tag{2.1.1}$$

The main result of this subsection is the following:

Theorem 2.1.1 *Let $A > 1$ be given. There is a zero measure subset $\mathcal{N}$ of $]0, +\infty[$ such that for any integers $0 \leq \ell \leq p+1$, any $m \in]0, +\infty[- \mathcal{N}$, there are constants $c > 0, N_0 \in \mathbb{N}$ such that the lower bound*

$$\begin{aligned} |F_m^\ell(n_0, \dots, n_{p+1})| \geq\ & c(1 + |n_0| + |n_{p+1}|)^{-d} (\log(e + |n_0| + |n_{p+1}|)^{-A} \\ & \times (1 + |n_0 - n_{p+1}|)^{-N_0} (1 + |n_1| + \cdots + |n_p|)^{-N_0} \end{aligned} \tag{2.1.2}$$

holds true for any $n_0, \dots, n_{p+1} \in \mathbb{Z}^d$ satisfying the following conditions:
- *If p is odd, or p is even and $\ell \neq p/2$, $|n_0|, |n_{p+1}| > \max(|n_1|, \dots, |n_p|)$,*
- *If p is even and $\ell = p/2$, $|n_0|, |n_{p+1}| > \max(|n_1|, \dots, |n_p|)$ and $|n_0| \neq |n_{p+1}|$.*

The proof of the theorem will rely on some geometric estimates that we shall deduce from results of [8]. Let $I \subset]0,+\infty[$ be some compact interval and define for $0 \le \ell \le p+1$ functions

$$\begin{aligned} f_\ell &: [0,1] \times [0,1]^{p+2} \times I \longrightarrow \mathbb{R} \\ &\quad (z, x_0, \dots, x_{p+1}, y) \to f_\ell(z, x_0, \dots, x_{p+1}, y) \\ g_\ell &: [0,1] \times [0,1]^p \times I \longrightarrow \mathbb{R}, \\ &\quad (z, x_1, \dots, x_p, y) \to g_\ell(z, x_1, \dots, x_p, y) \end{aligned} \tag{2.1.3}$$

by

$$\begin{aligned} f_\ell(z, x_0, \dots, x_{p+1}, y) &= \sum_{j=0}^{\ell} \sqrt{z^2 + y^2 x_j^2} - \sum_{j=\ell+1}^{p+1} \sqrt{z^2 + y^2 x_j^2} \\ g_\ell(z, x_1, \dots, x_p, y) &= z \Big[\sum_{j=1}^{\ell} \frac{z}{\sqrt{z^2 + y^2 x_j^2}} - \sum_{j=\ell+1}^{p} \frac{z}{\sqrt{z^2 + y^2 x_j^2}} \Big] \text{ when } z > 0, \\ g_\ell(0, x_1, \dots, x_p, y) &\equiv 0. \end{aligned} \tag{2.1.4}$$

Then the graphs of f_ℓ, g_ℓ are subanalytic subsets of $[0,1]^{p+3} \times I$ and $[0,1]^{p+1} \times I$ respectively, so that f_ℓ, g_ℓ are continuous subanalytic functions (see Bierstone-Milman [4] for an introduction to subanalytic sets and functions). Let us consider the set Γ of points $(z,x) \in [0,1]^{p+3}$ (resp. $(z,x) \in [0,1]^{p+1}$) such that $y \to f_\ell(z,x,y)$ (resp. $y \to g_\ell(z,x,y)$) vanishes identically. If $(z,x) \in \Gamma$ and $z \neq 0$, we must have

$$\ell = \frac{p}{2} \text{ and } \sum_{j \le \ell} x_j^{2\kappa} - \sum_{j \ge \ell+1} x_j^{2\kappa} = 0 \ \forall k \in \mathbb{N}^*$$

where the sum is taken respectively for $0 \le j \le p+1$ in the case of f_ℓ and $1 \le j \le p$ for g_ℓ. This implies that there is a bijection $\sigma : \{0, \dots, \ell\} \to \{\ell+1, \dots, p+1\}$ (resp. $\sigma : \{1, \dots, \ell\} \to \{\ell+1, \dots, p\}$) such that $x_{\sigma(j)} = x_j$ for any $j = 0, \dots, \ell$ (resp. $j = 1, \dots, \ell$) – see for instance the proof of lemma 5.6 in [8]. When p is even, denote by S_p the set of all bijections respectively from $\{0, \dots, p/2\}$ to $\{\frac{p}{2}+1, \dots, p+1\}$ and from $\{1, \dots, p/2\}$ to $\{\frac{p}{2}+1, \dots, p\}$. Define for $0 \le \ell \le p+1$

$$\begin{aligned} \rho_\ell(z,x) &\equiv z \text{ if } \ell \neq \frac{p}{2} \\ \rho_\ell(z,x) &= z \prod_{\sigma \in S_p} \big[\sum_{j \le p/2} (x_{\sigma(j)}^2 - x_j^2)^2 \big] \text{ if } \ell = \frac{p}{2}, \end{aligned} \tag{2.1.5}$$

where the sum in the above formula is taken for $j \ge 0$ (resp. $j \ge 1$) when we study f_ℓ (resp. g_ℓ). Then the set $\{\rho_\ell = 0\}$ contains those points (z,x) such that $y \to f_\ell(z,x,y)$ (resp. $y \to g_\ell(z,x,y)$) vanishes identically. Let us prove the following result:

Proposition 2.1.2 *(i) There are $\widetilde{N} \in \mathbb{N}, \alpha_0 > 0, \delta > 0, C > 0$ such that for any $0 \le \ell \le p+1$, any $\alpha \in]0, \alpha_0[$, any $N \ge \widetilde{N}$, any $(z,x) \in [0,1]^{p+3}$ (resp. $(z,x) \in [0,1]^{p+1}$) with $\rho_\ell(z,x) \neq 0$, the sets*

$$\begin{aligned} I_\ell^f(z,x,\alpha) &= \{y \in I; |f_\ell(z,x,y)| < \alpha \rho_\ell(z,x)^N\} \\ I_\ell^g(z,x,\alpha) &= \{y \in I; |g_\ell(z,x,y)| < \alpha \rho_\ell(z,x)^N\} \end{aligned} \tag{2.1.6}$$

have Lebesgue measure bounded from above by $C\alpha^{\delta}\rho_\ell(z,x)^{N\delta}$.

(ii) For any $N \geq \widetilde{N}$, *there is* $K \in \mathbb{N}$ *such that for any* $\alpha \in]0,\alpha_0[$, *any* $(z,x) \in [0,1]^{p+1}$, *the set* $I^g_\ell(z,x,\alpha)$ *may be written as the union of at most* K *open disjoint subintervals of* I.

Proof: (i) is nothing but the statement of theorem 5.1 in [8].

To prove (ii) we must show that $I^g_\ell(z,x,\alpha)$ has a number of connected components bounded from above by a fixed constant K. Let

$$\Sigma = \{(z,x,y,\alpha) \in [0,1]^{p+1} \times I \times [0,\alpha_0]; |g(z,x,\alpha)| < \alpha\rho(z,x)^N\}.$$

This is a relatively compact subanalytic subset of $\mathbb{R}^{p+3}$. Consider the projection

$$\begin{aligned}\pi : [0,1]^{p+1} \times I \times [0,\alpha_0] &\to [0,1]^{p+1} \times [0,\alpha_0]\\ (z,x,y,\alpha) &\to (z,x,\alpha).\end{aligned}$$

By theorem 2.5 of the paper of Hardt [11], the number of connected components of $\pi^{-1}(z,x,\alpha)\cap\Sigma$ is uniformly bounded. This concludes the proof. $\square$

We shall deduce theorem 2.1.1 from several lemmas. Let us first introduce some notations. When p is odd or p is even and $\ell \neq p/2$, we set $Z'^p_\ell = \emptyset$. When p is even and $\ell = p/2$, we define

$$\begin{aligned}Z'^p_\ell = \{n' = (n_1,\dots,n_p) \in (\mathbb{Z}^d)^p;\ &\text{there is a bijection } \sigma : \{1,\dots,\ell\} \to \{\ell+1,\dots,p\}\\ &\text{such that } |n_{\sigma(j)}| = |n_j|\ j = 1,\dots,\ell\}.\end{aligned} \tag{2.1.7}$$

We set also

$$Z^p_\ell = \{(n_0,n',n_{p+1}) \in (\mathbb{Z}^d)^{p+2}; n' \in Z'^p_\ell \text{ and } |n_0| = |n_{p+1}|\}. \tag{2.1.8}$$

Of course, $Z^p_\ell = \emptyset$ if p is odd or p is even and $\ell \neq p/2$.

We remark first that it is enough to prove (2.1.2) for those $(n_1,\dots,n_p)$ which do not belong to Z'^p_ℓ: actually, if p is even, $\ell = p/2$ and $(n_1,\dots,n_p) \in Z'^p_\ell$, we have $|F^\ell_m(n_0,\dots,n_{p+1})| = \left|\sqrt{m^2+|n_0|^2} - \sqrt{m^2+|n_{p+1}|^2}\right|$ which is bounded from below, when m stays in some compact interval, by

$$\frac{||n_0|^2 - |n_{p+1}|^2|}{\sqrt{m^2+|n_0|^2} + \sqrt{m^2+|n_{p+1}|^2}} \geq c(1+|n_0|+|n_{p+1}|)^{-1}$$

when $|n_0| \neq |n_{p+1}|, n_0, n_{p+1} \in \mathbb{Z}^d$. Consequently (2.1.2) holds true trivially. From now on, we shall always consider p-tuples n' which do not belong to Z'^p_ℓ.

Let us define for $\ell = 0,\dots,p+1$ another function on $(\mathbb{R}^d)^p$ given by

$$G^\ell_m(\xi_1,\dots,\xi_p) = \sum_{j=1}^{\ell}\sqrt{m^2+|\xi_j|^2} - \sum_{j=\ell+1}^{p}\sqrt{m^2+|\xi_j|^2}. \tag{2.1.9}$$

Let $J \subset]0,+\infty[$ be a given compact interval. For $\alpha > 0$, $N_0 \in \mathbb{N}, 0 \le \ell \le p+1$, $n = (n_0,\dots,n_{p+1}) \in (\mathbb{Z}^d)^{p+2}$ define

(2.1.10)
$$\begin{aligned} E_J^\ell(n,\alpha,N_0) = \{m \in J; |F_m^\ell(n_0,\dots,n_{p+1})| < \alpha(1+|n_0|+|n_{p+1}|)^{-d}(\log(e+|n_0|+|n_{p+1}|))^{-A} \\ \times(1+|n_0-n_{p+1}|)^{-N_0}(1+|n_1|+\cdots+|n_p|)^{-N_0}\}. \end{aligned}$$

We set also for $\beta > 0, N_1 \in \mathbb{N}^*, n' = (n_1,\dots,n_p) \in (\mathbb{Z}^d)^p - Z'^p_\ell$

$$E'^\ell_J(n',\beta,N_1) = \Big\{m \in J; \Big|\frac{\partial G_m^\ell}{\partial m}(n_1,\dots,n_p)\Big| < \beta(1+|n_1|+\cdots+|n_p|)^{-N_1}\Big\}. \tag{2.1.11}$$

We define for $\gamma > \beta$ a subset of $(\mathbb{Z}^d)^{p+2}$ by

$$\begin{aligned} S(\beta,\gamma,N_1) = \Big\{(n_0,\dots,n_{p+1}) \in (\mathbb{Z}^d)^{p+2} - Z_\ell^p; |n_0| < \frac{\gamma}{3\beta}(1+|n_1|+\cdots+|n_p|)^{N_1} \\ \text{or } |n_{p+1}| < \frac{\gamma}{3\beta}(1+|n_1|+\cdots+|n_p|)^{N_1}\Big\}. \end{aligned} \tag{2.1.12}$$

Lemma 2.1.3 *Let $\delta, \alpha_0, \widetilde{N}$ be the constants defined in the statement of proposition 2.1.2. There are constants $C_1 > 0$, $M \in \mathbb{N}^*$ such that for any $\beta \in]0,\alpha_0[$, any $N_1 \in \mathbb{N}$ with $N_1 \ge M\widetilde{N}$ and $N_1 > \frac{dpM}{\delta}$, one has*

$$\text{meas}\Big[\bigcup_{n' \in (\mathbb{Z}^d)^p - Z'^p_\ell} E'^\ell_J(n',\beta,N_1)\Big] \le C_1\beta^\delta. \tag{2.1.13}$$

Proof: Set $y = \frac{1}{m}$ and

$$z = \Big(1+\sum_{j=1}^p |n_j|\Big)^{-1}, \; x_j = |n_j|z, \quad j = 1,\dots,p.$$

Denote by X the set of points $(z,x) \in [0,1]^{p+1}$ of the preceding form for $(n_1,\dots,n_p)$ describing $(\mathbb{Z}^d)^p$. When p is even and $\ell = p/2$, let X'^p_ℓ be the image of Z'^p_ℓ defined by (2.1.7) under the map $n' \to (z,x)$. Using definition (2.1.5), we see that there are constants $M > 0, C > 0$, depending only on p, such that for $0 \le \ell \le p+1$

$$\forall (z,x) \in X - X'^p_\ell, \; z^M \le \rho_\ell(z,x) \le Cz. \tag{2.1.14}$$

Remark that

$$\begin{aligned} \frac{\partial G_m^\ell}{\partial m}(n') &= \sum_{j=1}^\ell \frac{m}{\sqrt{m^2+|n_j|^2}} - \sum_{j=\ell+1}^p \frac{m}{\sqrt{m^2+|n_j|^2}} \\ &= \frac{1}{z}g_\ell(z,x_1,\dots,x_p,y) \end{aligned}$$

with the above notations. Then if $I = \{m^{-1}; m \in J\}$, we see that $m \in E'^{\ell}_J(n', \beta, N_1)$ for $n' \notin Z'^p_\ell$ if and only if $y = \frac{1}{m}$ satisfies

$$|g_\ell(z, x_1, \ldots, x_p, y)| < \beta z^{N_1+1} \le \beta \rho_\ell(z,x)^{(N_1+1)/M} \tag{2.1.15}$$

using (2.1.14). Applying proposition 2.1.2 (i), we see that for any fixed value of $(z,x) \in X - X'^p_\ell$, the measure of those y such that (2.1.15) holds true is bounded from above by

$$C\beta^\delta \rho_\ell(z,x)^{\frac{N_1+1}{M}\delta} \le C\beta^\delta z^{\frac{N_1+1}{M}\delta}$$

if we assume $N_1 \ge M\widetilde{N}$ and $\beta \in]0, \alpha_0[$. Consequently, we get, with a constant C' depending only on J,

$$\mathrm{meas}\,(E'^{\ell}_J(n', \beta, N_1)) \le C'\beta^\delta (1 + |n_1| + \cdots + |n_p|)^{-\frac{N_1+1}{M}\delta}.$$

Inequality (2.1.13) follows from this estimate and the assumption on N_1. □

Lemma 2.1.4 *There are constants $M \in \mathbb{N}^*, \theta > 1, C_2 > 0$ such that for any $N_0, N_1 \in \mathbb{N}^*$ satisfying $N_0 > \widetilde{N}MN_1$ and $N_0\delta > d(p+2)MN_1$, any $0 < \beta < \gamma$ with $\frac{\gamma}{\beta} > \theta$, any $\alpha > 0$ satisfying $\alpha[\beta/2\gamma]^{-N_0/N_1} < \alpha_0$, one has*

$$\mathrm{meas}\Big[\bigcup_{n \in S(\beta,\gamma,N_1)} E^\ell_J(n, \alpha, N_0)\Big] \le C_2\alpha^\delta \Big(\frac{\beta}{2\gamma}\Big)^{-\frac{N_0}{N_1}\delta}. \tag{2.1.16}$$

Proof: We first remark that if $|n_0| + |n_{p+1}| > \frac{\gamma}{\beta}(1 + |n_1| + \cdots + |n_p|)^{N_1}$ and $n \in S(\beta, \gamma, N_1)$, then either

$$|n_0| \ge \frac{2}{3}\frac{\gamma}{\beta}(1 + |n_1| + \cdots + |n_p|)^{N_1} \text{ or } |n_{p+1}| \ge \frac{2}{3}\frac{\gamma}{\beta}(1 + |n_1| + \cdots + |n_p|)^{N_1}$$

which implies that

$$|F^\ell_m(n_0, \ldots, n_{p+1})| \ge c\frac{\gamma}{\beta}(1 + |n_1| + \cdots + |n_p|)^{N_1}$$

for some constant $c > 0$ depending only on p and J, if $\frac{\gamma}{\beta} > \theta$ large enough. Consequently, if $\alpha < \alpha_0$ small enough relatively to c, we see that we have in this case $E^\ell_J(n, \alpha, N_0) = \emptyset$ when $n \in S(\beta, \gamma, N_1)$. We may therefore consider only indices n such that

$$n \in S(\beta, \gamma, N_1) \text{ and } |n_0| + |n_{p+1}| \le \frac{\gamma}{\beta}(1 + |n_1| + \cdots + |n_p|)^{N_1}.$$

Consequently, for $m \in E^\ell_J(n, \alpha, N_0)$ and $n \in S(\beta, \gamma, N_1)$,we have

$$\begin{aligned} |F^\ell_m(n_0, \ldots, n_{p+1})| &< \alpha(1 + |n_1| + \cdots + |n_p|)^{-N_0} \\ &\le \alpha[\beta/2\gamma]^{-N_0/N_1}(1 + |n_0| + \cdots + |n_{p+1}|)^{-N_0/N_1}. \end{aligned} \tag{2.1.17}$$

Define for $n \in (\mathbb{Z}^d)^{p+2}$

$$z = \Big(1 + \sum_{j=0}^{p+1} |n_j|\Big)^{-1}, \; x_j = |n_j| z, \quad j = 0, \dots, p+1, \tag{2.1.18}$$

denote by $X \subset [0,1]^{p+3}$ the set of points (z, x) of the preceding form, and let X_ℓ^p be the image of the set Z_ℓ^p defined by (2.1.8) under the map $n \to (z, x)$. By (2.1.5) we have again

$$\forall (z, x) \in X - X_\ell^p, \; z^M \le \rho_\ell(z, x) \le Cz$$

for some large enough M, depending only on p. Moreover

$$F_m^\ell(n_0, \dots, n_{p+1}) = \frac{m}{z} f_\ell(z, x_0, \dots, x_{p+1}, y)$$

and (2.1.17) implies that if $n \in S(\beta, \gamma, N_1)$ and $m \in E_J^\ell(n, \alpha, N_0)$, then y satisfies

$$\begin{aligned} |f_\ell(z, x_0, \dots, x_{p+1}, y)| &\le C\alpha \Big(\frac{\beta}{2\gamma}\Big)^{-\frac{N_0}{N_1}} z^{1+\frac{N_0}{N_1}} \\ &\le C\alpha \Big(\frac{\beta}{2\gamma}\Big)^{-\frac{N_0}{N_1}} \rho_\ell(z, x)^{\frac{1}{M}\left(1+\frac{N_0}{N_1}\right)}. \end{aligned} \tag{2.1.19}$$

We assume that α, N_0, N_1 satisfy the conditions of the statement of the lemma. Then by (i) of proposition 2.1.2 we get that the measure of those $y \in I$ satisfying (2.1.19) is bounded from above by

$$C\Big[\alpha \Big(\frac{\beta}{2\gamma}\Big)^{-\frac{N_0}{N_1}}\Big]^\delta z^{\frac{\delta}{M}\left(1+\frac{N_0}{N_1}\right)}$$

for some constant C, independent of $N_0, N_1, \alpha, \beta, \gamma$. Consequently the measure of $E_J^\ell(n, \alpha, N_0)$ is bounded from above when $n \in S(\beta, \gamma, N_1)$ by

$$C\Big[\alpha \Big(\frac{\beta}{2\gamma}\Big)^{-\frac{N_0}{N_1}}\Big]^\delta (1 + |n_0| + \dots + |n_{p+1}|)^{-\frac{\delta}{M}\left(1+\frac{N_0}{N_1}\right)}.$$

The conclusion of the lemma follows by summation, using that $\frac{\delta}{M}(1 + \frac{N_0}{N_1}) > (p+2)d$. □

End of proof of theorem 2.1.1: We fix N_0, N_1 satisfying the conditions stated in lemmas 2.1.3 and 2.1.4, and such that $N_0 > dp + N_1$. We write when $n \notin S(\beta, \gamma, N_1)$, $0 \le \ell \le p+1$,

$$E_J^\ell(n, \alpha, N_0) \subset [E_J^\ell(n, \alpha, N_0) \cap E'^\ell_J(n', \beta, N_1)] \cup [E_J^\ell(n, \alpha, N_0) \cap (E'^\ell_J(n', \beta, N_1))^c]$$

and estimate, using that we reduced ourselves to those $n' \notin Z'^p_\ell$

(2.1.20)

$$\begin{aligned} \operatorname{meas}\Big[\bigcup_{n; n' \notin Z'^p_\ell} E_J^\ell(n, \alpha, N_0)\Big] \le \operatorname{meas}\Big[\bigcup_{n \in S(\beta, \gamma, N_1)} E_J^\ell(n, \alpha, N_0)\Big] + \operatorname{meas}\Big[\bigcup_{n' \notin Z'^p_\ell} E'^\ell_J(n', \beta, N_1)\Big] \\ + \operatorname{meas}\Big[\bigcup_{n \in S(\beta, \gamma, N_1)^c - Z_\ell^p} E_J^\ell(n, \alpha, N_0) \cap (E'^\ell_J(n', \beta, N_1))^c\Big]. \end{aligned}$$

Let us bound the measure of $E_J^\ell(n,\alpha,N_0)\cap(E'^\ell_J(n',\beta,N_1))^c$ for $n\in S(\beta,\gamma,N_1)^c - Z_\ell^p$. If m belongs to that set, the inequality in (2.1.10) holds true. Remark that we may assume $\ell\leq p$: if $\ell=p+1$, $|F_m^\ell(n_0,\dots,n_{p+1})|\geq c(1+|n_0|+|n_{p+1}|)$ for some $c>0$, which is not compatible with (2.1.10) for $\alpha<\alpha_0$ small enough. Let us write (2.1.10) as

$$\begin{aligned}||n_0|-|n_{p+1}|+\widetilde{G}_m(n_0,\dots,n_{p+1})|<\alpha(1+|n_0|+|n_{p+1}|)^{-d}(\log(e+|n_0|+|n_{p+1}|))^{-A}\\ \times(1+|n_0-n_{p+1}|)^{-N_0}(1+|n_1|+\dots+|n_p|)^{-N_0}\end{aligned} \tag{2.1.21}$$

with, using notation (2.1.9),

$$\begin{aligned}\widetilde{G}_m(n_0,\dots,n_{p+1})&=G_m(n_1,\dots,n_p)+R_m(n_0,n_{p+1})\\ R_m(n_0,n_{p+1})&=\sqrt{m^2+|n_0|^2}-|n_0|-\big(\sqrt{m^2+|n_{p+1}|^2}-|n_{p+1}|\big).\end{aligned} \tag{2.1.22}$$

Since $n\in S(\beta,\gamma,N_1)^c$, we have by (2.1.12)

$$|n_0|\geq\frac{\gamma}{3\beta}(1+|n_1|+\dots+|n_p|)^{N_1},\ \ |n_{p+1}|\geq\frac{\gamma}{3\beta}(1+|n_1|+\dots+|n_p|)^{N_1}. \tag{2.1.23}$$

Consequently, there is a constant $C>0$, depending only on J, such that

$$\Big|\frac{\partial R_m}{\partial m}(n_0,n_{p+1})\Big|\leq C\frac{\beta}{\gamma}(1+|n_1|+\dots+|n_p|)^{-N_1}.$$

If γ is large enough and $m\in {E'^\ell_J(n',\beta,N_1)}^c$, we deduce from (2.1.11) that

$$\Big|\frac{\partial\widetilde{G}_m}{\partial m}(n_0,n_{p+1})\Big|\geq\frac{\beta}{2}(1+|n_1|+\dots+|n_p|)^{-N_1}. \tag{2.1.24}$$

By (ii) of proposition 2.1.2, we know that there is $K\in\mathbb{N}$, independent of α,β,γ, such that the set $J-E'^\ell_J(n',\beta,N_1)$ is the union of at most K disjoint intervals $J_j(n',\beta,N_1), 1\leq j\leq K$. Consequently, we have

$$E_J^\ell(n,\alpha,N_0)\cap(E'^\ell_J(n',\beta,N_1))^c\subset\bigcup_{j=1}^K\{m\in J_j(n',\beta,N_1);\ (2.1.21)\text{ holds true}\}, \tag{2.1.25}$$

and on each interval $J_j(n',\beta,N_1)$, (2.1.24) holds true. We may on each such interval perform in the characteristic function of (2.1.21) the change of variable of integration given by $m\to\widetilde{G}_m(n_0,\dots,n_{p+1})$. Because of (2.1.24), this allows us to estimate the measure of (2.1.25) by

$$\begin{aligned}K\frac{2}{\beta}\alpha(1+|n_0|+|n_{p+1}|)^{-d}(\log(e+|n_0|+|n_{p+1}|))^{-A}\\ \times(1+|n_0-n_{p+1}|)^{-N_0}(1+|n_1|+\dots+|n_p|)^{-N_0+N_1}.\end{aligned}$$

Summing in $n_0,\dots,n_{p+1}$, we see that since $N_0>dp+N_1$ and $A>1$, the last term in (2.1.20) is bounded from above by $C_3\frac{\alpha}{\beta}$ with C_3 independent of α,β,γ. By lemmas 2.1.3 and 2.1.4, we may thus bound (2.1.20) by

$$C_2\alpha^\delta\Big(\frac{\beta}{2\gamma}\Big)^{-\frac{N_0}{N_1}\delta}+C_1\beta^\delta+C_3\frac{\alpha}{\beta}$$

if α, β are small enough, γ is large enough and $\alpha\big(\frac{\beta}{\gamma}\big)^{-\frac{N_0}{N_1}}$ is small enough. If we take $\beta = \alpha^\sigma, \gamma = \alpha^{-\sigma}$ with $\sigma > 0$ small enough, and $\alpha \ll 1$, we finally get for some $\delta' > 0$

$$\operatorname{meas}\Big[\bigcup_{n;n'\notin Z'^p_\ell} E^\ell_J(n,\alpha,N_0)\Big] \le C\alpha^{\delta'} \to 0 \text{ if } \alpha \to 0+.$$

This implies that the set of those $m \in J$ for which (2.1.2) does not hold true for any $c > 0$ is of zero measure, and concludes the proof of the theorem. □

In the following subsection, we shall also use a simpler version of theorem 2.1.1. Let us introduce some notations. For $m \in]0,+\infty[$, $\xi_j \in \mathbb{R}^d$, $j = 0,\dots,p+1$, $e = (e_0,\dots,e_{p+1}) \in \{-1,1\}^{p+2}$, define

$$\widetilde{F}^{(e)}_m(\xi_0,\dots,\xi_{p+1}) = \sum_{j=0}^{p+1} e_j\sqrt{m^2+|\xi_j|^2}. \tag{2.1.26}$$

When p is even and $\#\{j; e_j = 1\} = \frac{p}{2}+1$, denote by $Z^{(e)}$ the set of all $(n_0,\dots,n_{p+1}) \in (\mathbb{Z}^d)^{p+2}$ such that there is a bijection σ from $\{j; 0 \le j \le p+1, e_j = 1\}$ to $\{j; 0 \le j \le p+1, e_j = -1\}$ so that for any j in the first set $|n_j| = |n_{\sigma(j)}|$. In the other cases, set $Z^{(e)} = \emptyset$.

Proposition 2.1.5 *There is a zero measure subset $\mathcal{N}$ of $]0,+\infty[$, and for any $m \in]0,+\infty[-\mathcal{N}$, there are $N_0 \in \mathbb{N}, c > 0$ such that for any $(n_0,\dots,n_{p+1})$ in $(\mathbb{Z}^d)^{p+2} - Z^{(e)}$ one has*

$$|\widetilde{F}^{(e)}_m(n_0,\dots,n_{p+1})| \ge c(1+|n_0|+\dots+|n_{p+1}|)^{-N_0}. \tag{2.1.27}$$

Moreover, if $e_0e_{p+1} = 1$, one has the inequality

$$|\widetilde{F}^{(e)}_m(n_0,\dots,n_{p+1})| \ge c(1+|n_0|+|n_{p+1}|)(1+|n_1|+\dots+|n_p|)^{-N_0}. \tag{2.1.28}$$

Proof: The proof of (2.1.27) is similar to the one of lemma 2.1.4. Define

$$\tilde{f}^{(e)}(z,x_0,\dots,x_{p+1},y) = \sum_{j=0}^{p+1} e_j\sqrt{z^2+y^2x_j^2}$$

for $(z,x) \in [0,1]^{p+3}$, y belonging to some compact interval I of $]0,+\infty[$. Let X (resp. $X^{(e)}$) be the image of $(\mathbb{Z}^d)^{p+2}$ (resp. $Z^{(e)}$) under the map $(n_0,\dots,n_{p+1}) \to (z,x)$ given by (2.1.18). Using proposition 2.1.2, and reasoning as in the proof of lemma 2.1.4, one obtains that for large enough N_0 and small enough α, the measure of

$$\{y \in I; \exists (z,x) \in X - X^{(e)}, |\tilde{f}^{(e)}(z,x,y)| < \alpha z^{N_0+1}\} \tag{2.1.29}$$

is bounded from above by $C\alpha^\delta z^{\delta(N_0+1)}$ for some uniform constant $C > 0$ and $\delta > 0$. If N_0 is large enough, one deduces from this that the set of those m for which (2.1.27) does not hold true for any $c > 0$ is of zero measure.

To prove (2.1.28), remark that this inequality follows from (2.1.27) when there is some constant $C > 0$ such that $|n_0|+|n_{p+1}| \le C(1+|n_1|+\dots+|n_p|)$. On the other hand, when $|n_0|+|n_{p+1}| > C(1+|n_1|+\dots+|n_p|)$ with a large enough C, (2.1.28) is trivial because of the assumption $e_0e_{p+1} = 1$. This concludes the proof. □

2.2 Energy inequality and proof of the main theorem

We shall prove the main theorem, estimating for u solution of (1.3.3)

$$\Theta_s(u(t,\cdot)) = \frac{1}{2}\langle \Lambda_m^s u(t,\cdot), \Lambda_m^s u(t,\cdot)\rangle. \tag{2.2.1}$$

We compute first the time derivative of the above quantity.

Lemma 2.2.1 *There are $\nu \in \mathbb{R}_+, \delta > 0$ small enough and for any large enough $s_0 \in \mathbb{R}$, any $s \geq s_0$, there are:*
- *Multilinear operators $M_\ell^p \in \mathcal{M}_{p+1,\delta}^{2s-2,\nu}$ $\kappa \leq p \leq 2\kappa-1,\ 0 \leq \ell \leq p$, satisfying condition (1.2.17) for any p,*
- *Multilinear operators $\widetilde{M}_\ell^p \in \mathcal{M}_{p+1,\delta}^{2s-1,\nu}$ $\kappa \leq p \leq 2\kappa-1,\ 0 \leq \ell \leq p$,*
- *Elements $R^p \in \widetilde{\mathcal{R}}'^{2s,\nu}_{p+1}$ $\kappa \leq p \leq 2\kappa-1$,*
- *A map $u \to \widetilde{S}(u)$ defined on $H^s(\mathbb{T}^d, \mathbb{C})$ with values in $\mathbb{R}$, satisfying when $\|u\|_{H^{s_0}} \leq 1$*

$$|\widetilde{S}(u)| \leq C\|u\|_{H^{s_0}}^{2\kappa}\|u\|_{H^s}^2 \tag{2.2.2}$$

such that

(2.2.3)
$$\begin{aligned}\frac{d}{dt}\Theta_s(u(t,\cdot)) = \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\operatorname{Re} i\langle M_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle + \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\operatorname{Re} i\langle \widetilde{M}_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),\bar{u}\rangle \\ + \sum_{p=\kappa}^{2\kappa-1}\operatorname{Re} i\langle R^p(u,\bar{u}),u\rangle + \widetilde{S}(u).\end{aligned}$$

Proof: We compute using (1.3.3)

$$\begin{aligned}\frac{d}{dt}\Theta_s(u(t,\cdot)) &= \operatorname{Re}\langle \Lambda_m^s\partial_t u(t,\cdot), \Lambda_m^s u(t,\cdot)\rangle \\ &= \sum_{p=\kappa}^{2\kappa-1}\operatorname{Re} i\langle \Lambda_m^{2s}\operatorname{Op}(a^p(u,\bar{u};\cdot))u,u\rangle \\ &\quad + \sum_{p=\kappa}^{2\kappa-1}\operatorname{Re} i\langle \Lambda_m^{2s}\operatorname{Op}(a^p(u,\bar{u};\cdot))\bar{u},u\rangle \\ &\quad + \sum_{p=\kappa}^{2\kappa-1}\operatorname{Re} i\langle \Lambda_m^{2s}R^p(u),u\rangle + \operatorname{Re} i\langle \Lambda_m^s S(u), \Lambda_m^s u\rangle.\end{aligned} \tag{2.2.4}$$

The last term gives $\widetilde{S}(u)$. The first term in the right hand side may be written

$$\sum_{p=\kappa}^{2\kappa-1}\operatorname{Re}\frac{i}{2}\langle[\Lambda_m^{2s}\operatorname{Op}(a^p(u,\bar{u};\cdot)) - \operatorname{Op}(a^p(u,\bar{u};\cdot))^*\Lambda_m^{2s}]u,u\rangle$$

and so, since a^p is real valued and of order -1, gives according to proposition 1.2.7 the first sum in the right hand side of (2.2.3). Define $a^{p,\vee}(u,\bar{u};n) = a^p(u,\bar{u};-n)$. Since a^p is real valued, we may write the general term of the second sum in the right hand side of (2.2.4)

$$\begin{aligned}-\operatorname{Re} i\ \overline{\langle \Lambda_m^{2s}\operatorname{Op}(a^p(u,\bar{u};\cdot))\bar{u},u\rangle} &= -\operatorname{Re} i\langle \Lambda_m^{2s}\operatorname{Op}(a^{p,\vee}(u,\bar{u};\cdot))u,\bar{u}\rangle \\ &= \sum_{\ell=0}^{p}\operatorname{Re} i\langle \widetilde{M}_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),\bar{u}\rangle\end{aligned}$$

for some $\widetilde{M}_\ell^p \in \mathcal{M}_{p+1,\delta}^{2s-1,\nu}$, by lemma 1.2.6. This gives the second sum in the right hand side of (2.2.3). Finally, up to a change of notations, the last but one term in (2.2.4) gives the last but one term in (2.2.3). This concludes the proof. $\square$

Our aim now is to perturb $\Theta_s(u(t,\cdot))$ in such a way that terms homogeneous of degree smaller than $2\kappa+2$ will be eliminated in the right hand side of (2.2.3).

Proposition 2.2.2 *There is s_0 large enough and for any $s > s_0$ two maps*

$$\begin{aligned}\Theta_s^1 : H^s(\mathbb{T}^d,\mathbb{C})\times]0,1/2[&\to \mathbb{R}\\ (u,\epsilon) &\to \Theta_s^1(u,\epsilon)\\ \Theta_s^2 : H^s(\mathbb{T}^d,\mathbb{C}) &\to \mathbb{R}\\ u &\to \Theta_s^2(u),\end{aligned} \tag{2.2.5}$$

such that there is a constant $C_s > 0$ and for any $u \in H^s(\mathbb{T}^2,\mathbb{C})$ with $\|u\|_{H^{s_0}} \le 1$, any $\epsilon \in]0,1/2[$

$$\begin{aligned}|\Theta_s^1(u,\epsilon)| &\le C_s|\log\epsilon|^A\epsilon^{-\kappa\left(1-\frac{2}{d}\right)}\|u\|_{H_0^s}^{\kappa}\|u\|_{H^s}^2\\ |\Theta_s^2(u)| &\le C_s\|u\|_{H_0^s}^{\kappa}\|u\|_{H^s}^2,\end{aligned} \tag{2.2.6}$$

and such that

$$R(u) \stackrel{\text{def}}{=} \frac{d}{dt}[\Theta_s(u(t,\cdot)) - \Theta_s^1(u(t,\cdot),\epsilon) - \Theta_s^2(u(t,\cdot))] \tag{2.2.7}$$

satisfies for any $\epsilon \in]0,1/2[$

$$|R(u)| \le C_s|\log\epsilon|^A\epsilon^{-\kappa\left(1-\frac{2}{d}\right)}\|u\|_{H^{s_0}}^{2\kappa}\|u\|_{H^s}^2 + C_s\epsilon^{\frac{2\kappa}{d}}\|u\|_{H^{s_0}}^{\kappa}\|u\|_{H^s}^2. \tag{2.2.8}$$

To prove proposition 2.2.2, we shall need the following lemma. When $M(u_1,\dots,u_{p+1})$ is a $(p+1)$-linear form, let us define for $0 \le \ell \le p+1$

$$\begin{aligned}L_\ell^{\pm}(M)(u_1,\dots,u_{p+1}) = \pm\Lambda_m M(u_1,\dots,u_{p+1}) &- \sum_{j=1}^{\ell} M(u_1,\dots,\Lambda_m u_j,\dots,u_{p+1})\\ &+ \sum_{j=\ell+1}^{p+1} M(u_1,\dots,\Lambda_m u_j,\dots,u_{p+1}).\end{aligned} \tag{2.2.9}$$

Lemma 2.2.3 *Let $\mathcal{N}$ be the subset defined in theorem 2.1.1, and fix $m \in]0, +\infty[- \mathcal{N}$. There is $\bar\nu$ such that the following statements hold true for any $\delta > 0$ small enough, any large enough s, any integer p with $\kappa \le p \le 2\kappa - 1$, any integer ℓ with $0 \le \ell \le p+1$:*

(i) If M^p_ℓ is an element of $\mathcal{M}^{2s-2,\nu}_{p+1,\delta}$, define

(2.2.10)
$$M^{p,\epsilon}_\ell(u_1,\dots,u_{p+1}) = \sum_{n_0}\sum_{n_{p+1}} \mathbf{1}_{\{|n_0|+|n_{p+1}|<\epsilon^{-\kappa/d}, |n_0|\neq|n_{p+1}|\}} \Pi_{n_0} M^p_\ell(u_1,\dots,u_p,\Pi_{n_{p+1}}u_{p+1}).$$

Then there is $\underline{M}^{p,\epsilon}_\ell \in \mathcal{M}^{2s,\nu+\bar\nu}_{p+1,\delta}$ satisfying

$$L^-_\ell(\underline{M}^{p,\epsilon}_\ell)(u_1,\dots,u_{p+1}) = M^{p,\epsilon}_\ell(u_1,\dots,u_{p+1}) \tag{2.2.11}$$

with the estimate for all $N \ge \bar\nu$

$$\|\underline{M}^{p,\epsilon}_\ell\|_{\mathcal{M}^{2s,\nu+\bar\nu}_{p+1,\delta}(N-\bar\nu)} \le C|\log\epsilon|^A \epsilon^{-\kappa\left(1-\frac{2}{d}\right)} \|M^p_\ell\|_{\mathcal{M}^{2s-2,\nu}_{p+1,\delta}(N)}. \tag{2.2.12}$$

(ii) Let $\widetilde{M}^p_\ell \in \mathcal{M}^{2s-1,\nu}_{p+1,\delta}$. There is $\underline{\widetilde{M}}^p_\ell \in \mathcal{M}^{2s-2,\nu+\bar\nu}_{p+1,\delta}$ with

$$L^+_\ell(\underline{\widetilde{M}}^p_\ell)(u_1,\dots,u_{p+1}) = \widetilde{M}^p_\ell(u_1,\dots,u_{p+1}). \tag{2.2.13}$$

(iii) Let $R^p_\ell \in \mathcal{R}^{2s,\nu}_{p+1}$. Assume that for any $(n_0,\dots,n_{p+1}) \in Z^{(e)}$ defined after (2.1.26) with $e_0 = \dots = e_\ell = -1$, $e_{\ell+1} = \dots = e_{p+1} = 1$, any $u_1,\dots,u_{p+1}$

$$\Pi_{n_0} R^p_\ell(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}) \equiv 0. \tag{2.2.14}$$

Then there is $\underline{R}^p_\ell \in \mathcal{R}^{2s,\nu+\bar\nu}_{p+1}$ such that

$$L^-_\ell(\underline{R}^p_\ell)(u_1,\dots,u_{p+1}) = R^p_\ell(u_1,\dots,u_{p+1}). \tag{2.2.15}$$

Proof: (i) We substitute in (2.2.11) $\Pi_{n_j}u_j$ to u_j $j = 1,\dots,p+1$, and compose on the left with Π_{n_0}. According to (2.2.9) and using notations (2.1.1), equality (2.2.11) may be written

(2.2.16)
$$-F^\ell_m(n_0,\dots,n_{p+1})\Pi_{n_0}\underline{M}^{p,\epsilon}_\ell(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}) = \Pi_{n_0}M^{p,\epsilon}_\ell(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}).$$

We may use (2.1.2) to bound $|F^\ell_m(n_0,\dots,n_{p+1})|$ from below, since the assumption concerning $(n_0,\dots,n_{p+1})$ of theorem 2.1.1 holds true because of condition (1.2.8) of definition 1.2.2, and because of the cut-off for $|n_0| \neq |n_{p+1}|$ in (2.2.10). We deduce from (2.1.2) and the condition $|n_0| + |n_{p+1}| \le \epsilon^{-\kappa/d}$ the estimate

$$\begin{aligned} |F^\ell_m(n_0,\dots,n_{p+1})|^{-1} &\le C(1+|n_0|+|n_{p+1}|)^2 \epsilon^{-\kappa\left(1-\frac{2}{d}\right)} |\log\epsilon|^A \\ &\quad \times (1+|n_0-n_{p+1}|)^{N_0}(1+|n_1|+\dots+|n_p|)^{N_0}. \end{aligned} \tag{2.2.17}$$

If we define

$$\underline{M}^{p,\epsilon}_\ell(u_1,\dots,u_p) = -\sum_{n_0}\cdots\sum_{n_{p+1}} F^\ell_m(n_0,\dots,n_{p+1})^{-1} \Pi_{n_0} M^{p,\epsilon}_\ell(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})$$

we obtain according to (2.2.17) and definition 1.2.2, that $\underline{M}_\ell^{p,\epsilon} \in \mathcal{M}_{p+1,\delta}^{2s,\nu+2N_0}$ with the estimate (2.2.12) with $\bar{\nu} = 2N_0$. This gives (i) of the lemma.

(ii) In the same way as above, we deduce from (2.2.13) and (2.2.9) the equality

$$\widetilde{F}_m^{(e)}(n_0, \dots, n_{p+1})\Pi_{n_0}\underline{\widetilde{M}}_\ell^p(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1}) = \Pi_{n_0}\widetilde{M}_\ell^p(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1}) \tag{2.2.18}$$

where $\widetilde{F}_m^{(e)}$ is defined by (2.1.26) with $e_1 = \dots = e_\ell = -1$, $e_0 = e_{\ell+1} = \dots = e_{p+1} = 1$. Remark that we may assume that $(n_0, \dots, n_{p+1}) \notin Z^{(e)}$ defined after (2.1.26). Actually, because of the support condition (1.2.8), we cannot find any $j \in \{1, \dots, \ell\}$ such that $|n_j| = |n_0|$. Consequently, we may use the lower bound (2.1.28). If we define $\underline{\widetilde{M}}_\ell^p$ dividing in (2.2.18) by $\widetilde{F}_m^{(e)}$, we thus see that we get an element of $\mathcal{M}_{p+1,\delta}^{2s-2,\nu+\bar{\nu}}$ for some $\bar{\nu}$. This gives (ii).

(iii) We deduce again from (2.2.15)

$$\widetilde{F}_m^{(e)}(n_0, \dots, n_{p+1})\Pi_{n_0}\underline{R}_\ell^p(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1}) = \Pi_{n_0}R_\ell^p(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1})$$

where $e_0 = \dots = e_\ell = -1$, $e_{\ell+1} = \dots = e_{p+1} = 1$. By condition (2.2.14) we may assume that $(n_0, \dots, n_{p+1}) \notin Z^{(e)}$. Using (2.1.27), we deduce from the above equality and definition 1.2.8 that $\underline{R}_\ell^p \in \mathcal{R}_{p+1}^{2s,\nu+\bar{\nu}}$. This concludes the proof. $\square$

Proof of proposition 2.2.2: Consider the right hand side of (2.2.3). Since M_ℓ^p satisfies conditions (1.2.17), we may always assume that $\Pi_{n_0}M_\ell^p(u_1, \dots, u_p, \Pi_{n_{p+1}}u_{p+1}) \equiv 0$ for any $u_1, \dots, u_{p+1}$ and any n_0, n_{p+1} with $|n_0| = |n_{p+1}|$. Let us then decompose

$$M_\ell^p(u_1, \dots, u_{p+1}) = M_\ell^{p,\epsilon}(u_1, \dots, u_{p+1}) + V_\ell^{p,\epsilon}(u_1, \dots, u_{p+1})$$

where the first term is given by (2.2.10) and the second one by

$$V_\ell^{p,\epsilon}(u_1, \dots, u_{p+1}) = \sum_{n_0}\sum_{n_{p+1}} \mathbf{1}_{\{|n_0|+|n_{p+1}| \geq \epsilon^{-\kappa/d}\}}\Pi_{n_0}M_\ell^p(u_1, \dots, u_p, \Pi_{n_{p+1}}u_{p+1}). \tag{2.2.19}$$

By (1.2.9), we get

(2.2.20)

$$\begin{aligned}\|\Pi_{n_0}V_\ell^{p,\epsilon}(u_1, \dots, u_{p+1})\|_{H^{-s}} \leq C_N \sum_{n_1} \cdots \sum_{n_{p+1}} &(1 + |n_0| + |n_{p+1}|)^{2s-2}\frac{(1 + |n'|)^{\nu+N}}{(|n_0 - n_{p+1}| + |n'| + 1)^N} \\ &\times \mathbf{1}_{\{|n_0|+|n_{p+1}| \geq \epsilon^{-\kappa/d}, |n_0 - n_{p+1}| < \delta(|n_0|+|n_{p+1}|), |n'| < \delta(|n_0|+|n_{p+1}|)\}} \\ &\times (1 + |n_0|)^{-s}(1 + |n_{p+1}|)^{-s}\prod_1^p(1 + |n_j|)^{-s_0}c_{n_{p+1}}\prod_1^p\|u_j\|_{H^{s_0}}\|u_{p+1}\|_{H^s}\end{aligned}$$

for a sequence $(c_{n_{p+1}})_{n_{p+1}}$ in the unit ball of ℓ^2. The gain of two powers of $(|n_0| + |n_{p+1}|)$ in the first term in the right hand side, coming from the fact that $M_\ell^p \in \mathcal{M}_{p+1,\delta}^{2s-2,\nu}$, together with the condition $|n_0| + |n_{p+1}| > \epsilon^{-\kappa/d}$, allows us to estimate, for N large enough and s_0

large enough with respect to ν, (2.2.20) by $C\epsilon^{2\kappa/d}\prod_1^p\|u_j\|_{H^{s_0}}\|u_{p+1}\|_{H^s}c'_{n_0}$ for a new ℓ^2-sequence $(c'_{n_0})_{n_0}$. Consequently, the quantity

$$\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\operatorname{Re} i\langle V_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u,\dots,u),u\rangle \tag{2.2.21}$$

is bounded from above by the last term in the right hand side of (2.2.8). In the rest of the proof, we may therefore replace in the right hand side of (2.2.3) M_ℓ^p by $M_\ell^{p,\epsilon}$.

The contributions $R^p \in \widetilde{\mathcal{R}}'^{2s,\nu}_{p+1}$ in the right hand side of (2.2.3) satisfy condition (1.2.28). Consequently, we may assume that (2.2.14) holds true for R_ℓ^p. Apply now (i) (resp. (ii), resp. (iii)) of lemma 2.2.3 to $M_\ell^{p,\epsilon}$ (resp. $\widetilde{M}_\ell^p$, resp. R_ℓ^p). This defines $\underline{M}_\ell^{p,\epsilon}$ (resp. $\underline{\widetilde{M}}_\ell^p$, resp. $\underline{R}_\ell^p$). Set

(2.2.22)

$$\begin{aligned}
\Theta_s^1(u,\epsilon) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\operatorname{Re}\langle \underline{M}_\ell^{p,\epsilon}(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle\\
\Theta_s^2(u) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\operatorname{Re}\langle \underline{\widetilde{M}}_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),\bar{u}\rangle + \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\operatorname{Re}\langle \underline{R}_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle.
\end{aligned}$$

The general term in $\Theta_s^1(u,\epsilon)$ has modulus bounded from above by

$$\|\underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u\dots,u)\|_{H^{-s}}\|u\|_{H^s} \le C\epsilon^{-\kappa\left(1-\frac{2}{d}\right)}|\log\epsilon|^A\|u\|_{H^{s_0}}^\kappa\|u\|_{H^s}^2$$

for u in the unit ball of H^{s_0}, using lemma 1.2.3 and (2.2.12). This gives the first inequality (2.2.6). To obtain the estimate of Θ_s^2 given in (2.2.6), we apply to $\underline{\widetilde{M}}_\ell^p$ (resp. $\underline{R}_\ell^p$) lemma 1.2.3 (resp. lemma 1.2.9), remarking that if in (1.2.25) $\mu = 2s$ and s_0 is large enough, the left hand side controls the H^{-s_0} norm of $\underline{R}_\ell^p(\bar{u},\dots,\bar{u},u\dots,u)$. Consequently, we are left with proving (2.2.8). Compute using (1.3.2) and notation (2.2.9)

$$\begin{aligned}
\frac{d}{dt}\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u\dots,u),u\rangle &= i\langle L_\ell^-(\underline{M}_\ell^{p,\epsilon})(\bar{u},\dots,\bar{u},u\dots,u),u\rangle\\
&\quad+\sum_{j=1}^{\ell} i\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{F},\dots,\bar{u},u\dots,u),u\rangle\\
&\quad-\sum_{j=\ell+1}^{p+1} i\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u\dots,F,\dots,u),u\rangle\\
&\quad+ i\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u\dots,u),F\rangle.
\end{aligned}\tag{2.2.23}$$

By assumption on F, we have $\|F(\cdot,v)\|_{H^s} \le C\|u\|_{H^{s_0}}^\kappa\|u\|_{H^s}$ if $s \ge s_0$ large enough and $\|u\|_{H^{s_0}} \le 1$. If we apply lemma 1.2.3 and (2.2.12), we see that the last three terms in (2.2.23) have modulus bounded from above by the first term in the right hand side of (2.2.8). When computing $\frac{d}{dt}\Theta_s(u(t,\cdot))$, the first term in the right hand side of (2.2.3) is the sum of (2.2.21), which has been seen to be controlled by the second term in the right hand side of (2.2.8), and of the real part of the first term in the right hand side of (2.2.23), because of (2.2.11). Consequently, these contributions will cancel out each other in the expression $\frac{d}{dt}[\Theta_s(u(t,\cdot)) - \Theta_s^1(u(t,\cdot),\epsilon)]$. It remains to prove that in $\frac{d}{dt}[\Theta_s(u(t,\cdot)) - \Theta_s^1(u(t,\cdot),\epsilon) - \Theta_s^2(u(t,\cdot))]$, the second and third term in the right hand side of (2.2.3) are canceled, up to remainders estimated by (2.2.8). Using again (1.3.2) we get

$$\begin{aligned}\frac{d}{dt}\langle \underline{\widetilde{M}}_\ell^p(\bar u,\dots,\bar u,u\dots,u),\bar u\rangle &= i\langle L_\ell^+(\underline{\widetilde{M}}_\ell^p)(\bar u,\dots,\bar u,u\dots,u),\bar u\rangle\\ &\quad+\sum_{j=1}^{\ell} i\langle \underline{\widetilde{M}}_\ell^p(\bar u,\dots,\bar F,\dots,\bar u,u\dots,u),\bar u\rangle\\ &\quad-\sum_{j=\ell+1}^{p+1} i\langle \underline{\widetilde{M}}_\ell^p(\bar u,\dots,\bar u,u\dots,F,\dots,u),\bar u\rangle\\ &\quad- i\langle \underline{\widetilde{M}}_\ell^p(\bar u,\dots,\bar u,u\dots,u),\bar F\rangle.\end{aligned}$$

The last three terms are bounded by $C\|u\|_{H^{s_0}}^{2\kappa}\|u\|_{H^s}^2$ by lemma 1.2.3, so by the first term in the right hand side of (2.2.8). By (2.2.13), the first term in the right hand side will cancel the $\widetilde{M}_\ell^p$ contributions to (2.2.3). Finally we compute

$$\begin{aligned}\frac{d}{dt}\langle \underline{R}_\ell^p(\bar u,\dots,\bar u,u\dots,u),u\rangle &= i\langle L_\ell^-(\underline{R}_\ell^p)(\bar u,\dots,\bar u,u\dots,u),u\rangle\\ &\quad+ i\sum_{j=1}^{\ell}\langle \underline{R}_\ell^p(\bar u,\dots,\bar F,\dots,\bar u,u\dots,u),u\rangle\\ &\quad- i\sum_{j=\ell+1}^{p+1}\langle \underline{R}_\ell^p(\bar u,\dots,\bar u,u\dots,F,\dots,u),u\rangle\\ &\quad+ i\langle \underline{R}_\ell^p(\bar u,\dots,\bar u,u\dots,u),F\rangle.\end{aligned}$$

By lemma 1.2.9, the last three terms are estimated by the right hand side of (2.2.8). The first one, according to (2.2.15), cancels the contribution of R_ℓ^p in (2.2.3) when computing (2.2.7). This concludes the proof of the proposition. □

Proof of theorem 1.1.1: We deduce from (2.2.7), (2.2.8)

$$\begin{aligned}\Theta_s(u(t,\cdot)) &\le \Theta_s(u(0,\cdot)) - \Theta_s^1(u(0,\cdot),\epsilon) - \Theta_s^2(u(0,\cdot)) + \Theta_s^1(u(t,\cdot),\epsilon) + \Theta_s^2(u(t,\cdot))\\ &\quad+ C_s|\log\epsilon|^A\epsilon^{-\kappa\left(1-\frac{2}{d}\right)}\int_0^t \|u(\tau,\cdot)\|_{H^{s_0}}^{2\kappa}\|u(\tau,\cdot)\|_{H^s}^2\,d\tau\\ &\quad+ C_s\epsilon^{2\kappa/d}\int_0^t \|u(\tau,\cdot)\|_{H^{s_0}}^{\kappa}\|u(\tau,\cdot)\|_{H^s}^2\,d\tau.\end{aligned}\tag{2.2.24}$$

Take $B > 1$ a constant such that for any (v_0, v_1) in the unit ball of $H^{s+1}\times H^s$, $u(0,\cdot) = \epsilon(-iv_1 + \Lambda_m v_0)$ satisfies $\|u(0,\cdot)\|_{H^s} \le B\epsilon$. Let $K > B$ be another constant to be chosen, and assume that for τ in some interval $[0,T]$ we have $\|u(\tau,\cdot)\|_{H^s} \le K\epsilon$ and $\|u(\tau,\cdot)\|_{H^{s_0}} \le 1$. Using (2.2.1) and (2.2.6) we deduce from (2.2.24) that there is a constant $C > 0$, independent of B, K, ϵ, such that as long as $t \in [0,T]$

$$\|u(t,\cdot)\|_{H^s}^2 \le C[B^2 + |\log\epsilon|^A\epsilon^{2\kappa/d}K^{\kappa+2} + |\log\epsilon|^A\epsilon^{\kappa\left(1+\frac{2}{d}\right)}tK^{2\kappa+2}]\epsilon^2.$$

If we assume that $T \leq c\epsilon^{-\kappa\left(1+\frac{2}{d}\right)}|\log \epsilon|^{-A}$ for a small enough $c > 0$, and that ϵ is small enough, we get $\|u(t,\cdot)\|^2_{H^s} \leq C(2B^2)\epsilon^2$. If K has been chosen initially so that $2CB^2 < K^2$, we get by a standard continuity argument that the *a priori* bound $\|u(t,\cdot)\|_{H^s} \leq K\epsilon$ holds true on $[0, c\epsilon^{-\kappa\left(1+\frac{2}{d}\right)}|\log \epsilon|^{-A}]$, and so that the solution extends to such an interval. This concludes the proof of the theorem. $\square$

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